Difference Equations
to
Differential Equations

Section 8.5

Applications: Pendulums and Mass-Spring Systems

In this section we will investigate two applications of our work in Section 8.4. First, we will consider the motion of a pendulum, a problem originally mentioned in Section 2.2 in connection with the trigonometric functions. Second, we will discuss the motion of an object vibrating at the end of a spring.

The motion of a pendulum

Consider a pendulum consisting of a bob of mass m at the end of a rigid rod of length b. We will assume that the mass of the rod is negligible in comparison with the mass of the bob. Let $x(t)$ be the angle between the rod and the vertical at time t, with $x(t) > 0$ for angles measured in the counterclockwise direction and $x(t) < 0$ for angles measured in the clockwise direction. See Figure 8.5.1. Suppose the bob is pulled through an angle α and then released. That is, suppose our initial conditions are $x(0) = \alpha$ and $\dot{x}(0) = 0$. If we view the motion of the pendulum in the complex plane, with the real axis vertical, positive direction downward, and the imaginary axis horizontal, positive direction to the right, then the position of the bob at time t is given by

$$z(t) = be^{ix(t)}. \tag{8.5.1}$$

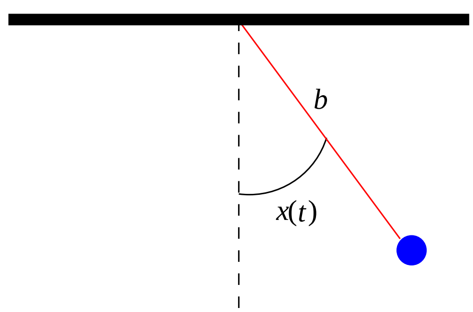

Figure 8.5.1 A pendulum

Then we have

$$\dot{z} = ib\dot{x}e^{ix} \tag{8.5.2}$$

and

$$\begin{aligned}\ddot{z} &= -b\dot{x}^2e^{ix} + ib\ddot{x}e^{ix} \\ &= -b\dot{x}^2(\cos(x) + i\sin(x)) + ib\ddot{x}(\cos(x) + i\sin(x)) \\ &= (-b\dot{x}^2\cos(x) - b\ddot{x}\sin(x)) + i(-b\dot{x}^2\sin(x) + b\ddot{x}\cos(x)).\end{aligned} \tag{8.5.3}$$

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Now $\ddot{z}$ is the acceleration of the pendulum, and so $m\ddot{z}$ must be equal to the force of gravity acting on the bob, namely, a force of magnitude mg acting in the downward direction, the direction of the positive real axis. Hence we must have $g = \ddot{z}$, that is,

$$g = (-b\dot{x}^2\cos(x) - b\ddot{x}\sin(x)) + i(-b\dot{x}^2\sin(x) + b\ddot{x}\cos(x)). \tag{8.5.4}$$

Equating the real and imaginary parts of the two sides of (8.5.4) gives us

$$g = -b\dot{x}^2\cos(x) - b\ddot{x}\sin(x) \tag{8.5.5}$$

and

$$0 = -b\dot{x}^2\sin(x) + b\ddot{x}\cos(x). \tag{8.5.6}$$

Multiplying (8.5.5) by $-\sin(x)$ and (8.5.6) by $\cos(x)$ gives us

$$-g\sin(x) = b\dot{x}^2\cos(x)\sin(x) + b\ddot{x}\sin^2(x) \tag{8.5.7}$$

and

$$0 = -b\dot{x}^2\sin(x)\cos(x) + b\ddot{x}\cos^2(x). \tag{8.5.8}$$

Adding (8.5.7) and (8.5.8) together yields

$$-g\sin(x) = b\ddot{x}(\sin^2(x) + \cos^2(x)) = b\ddot{x}. \tag{8.5.9}$$

Thus

$$\ddot{x} = -\frac{g}{b}\sin(x). \tag{8.5.10}$$

So we have reduced the problem of describing the motion of the pendulum to the problem of solving the second order differential equation (8.5.10) subject to the initial conditions $x(0) = \alpha$ and $\dot{x}(0) = 0$. Unfortunately, this equation is not linear. In fact, it is not possible to find a closed form solution for this equation. In Section 8.6 we will discuss how to study this equation using numerical approximations, but for now we will take a different approach to finding an approximate solution. Since we know

$$\sin(x) = x + o(x) \tag{8.5.11}$$

from our work on best affine approximations in Chapter 2, it is reasonable to replace $\sin(x)$ by x for small values of x. Hence, if we restrict to the case where α is small, we may replace (8.5.10) by the linear equation

$$\ddot{x} = -\frac{g}{b}x. \tag{8.5.12}$$

Since this equation is homogeneous with constant coefficients, we may solve it using the techniques of Section 8.4. Specifically, the characteristic equation for this equation is

$$k^2 + \frac{g}{b} = 0, \tag{8.5.13}$$

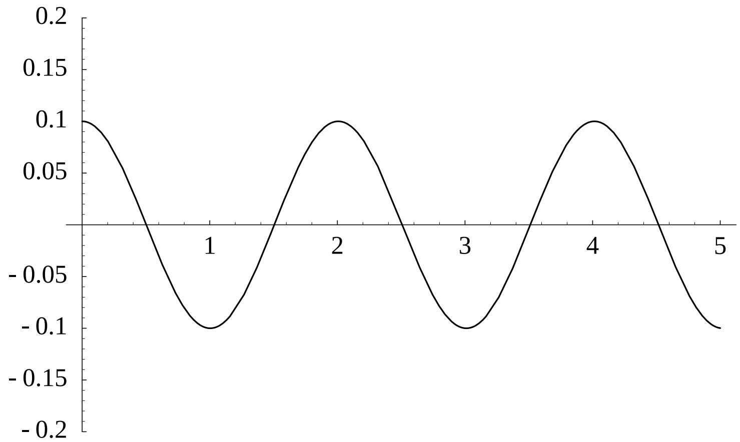

Figure 8.5.2 Motion of a pendulum

which has roots

$$k_1 = -i\sqrt{\frac{g}{b}} \tag{8.5.14}$$

and

$$k_2 = i\sqrt{\frac{g}{b}} \tag{8.5.15}$$

Hence the general solution is

$$x = c_1 \cos\left(\sqrt{\frac{g}{b}}t\right) + c_2 \sin\left(\sqrt{\frac{g}{b}}t\right). \tag{8.5.16}$$

Then

$$\dot{x} = -c_1\sqrt{\frac{g}{b}} \sin\left(\sqrt{\frac{g}{b}}t\right) + c_2\sqrt{\frac{g}{b}} \cos\left(\sqrt{\frac{g}{b}}t\right), \tag{8.5.17}$$

and so $x(0) = c_1$ and $\dot{x}(0) = c_2\sqrt{\frac{g}{b}}$. Hence the initial conditions $x(0) = \alpha$ and $\dot{x}(0) = 0$ imply $c_1 = \alpha$ and $c_2 = 0$. Thus

$$x = \alpha \cos\left(\sqrt{\frac{g}{b}}t\right) \tag{8.5.18}$$

The graph of x for the case $b = 1$ meter and $\alpha = 0.1$ radians, in which case we use $g = 9.8$ meters per second per second, is shown in Figure 8.5.2.

One consequence of (8.5.18) is that the period of the motion, that is, the time it takes the bob to make one complete oscillation, is

$$\frac{2\pi}{\sqrt{\frac{g}{b}}} = 2\pi\sqrt{\frac{b}{g}}, \tag{8.5.19}$$

independent of the value of α. Of course, we are working under the approximation $\sin(x) \approx x$, so (8.5.19) is actually only an approximation of the period. Nevertheless, the approximation is very good for small oscillations and is the reason pendulums were used to measure time in early clocks.

Vibrations in mechanical systems: mass-spring systems

In this example we consider the motion of an object of mass m suspended on a spring, as shown in Figure 8.5.3. We will measure the position of the object along a vertical axis, with the equilibrium position at 0 and the positive direction downward. Let $x(t)$ denote the position of the object at time t and suppose the object is released from rest at position x_0. That is, we suppose that $x(0) = x_0$ and $\dot{x}(0) = 0$. If we ignore any damping forces, such as resistance to the motion due to the surrounding medium, such as air or oil, then the only forces acting on the object are the force of gravity, contributing a term of mg, and the restorative force of the spring, given, according to Hooke's law, by $k\ell$ for some constant $k > 0$, where ℓ is the amount the spring is stretched or compressed from its natural length. If we let $\Delta\ell$ be the amount the spring is stretched when the object is at the equilibrium position, that is, when $x = 0$, then at any time the spring is stretched or compressed by $x + \Delta\ell$. Thus at any time t the force acting on the object is

$$F = mg - k(x + \Delta\ell). \tag{8.5.20}$$

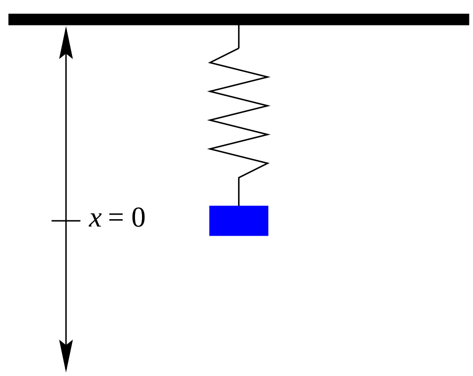

Figure 8.5.3 Mass on a spring at equilibrium

In particular, if the object is at rest at its equilibrium position, then both $x = 0$ and $F = 0$. Hence

$$0 = mg - k\Delta\ell, \tag{8.5.21}$$

and so

$$mg = k\Delta\ell. \tag{8.5.22}$$

Thus (8.5.20) simplifies to $F = -kx$. Applying Newton's second law of motion, we have

$$m\ddot{x} = -kx, \tag{8.5.23}$$

from which we obtain

$$\ddot{x} = -\frac{k}{m}x. \tag{8.5.24}$$

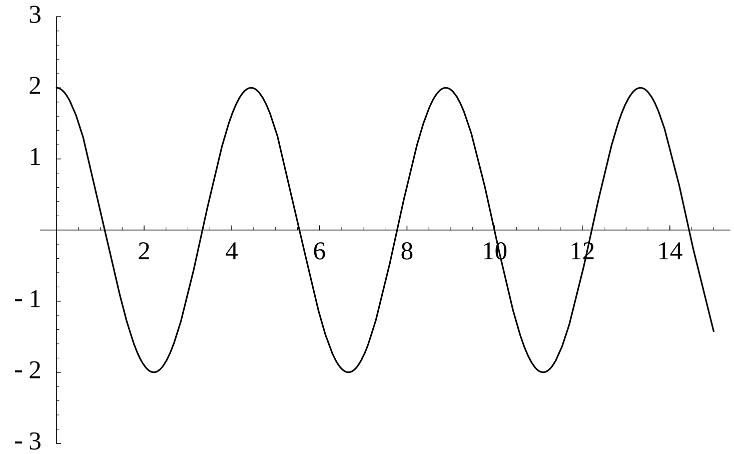

Figure 8.5.4 Motion of a mass-spring system without damping

This equation is of the same form as the equation derived above for approximating the motion of a pendulum. Hence, using the same reasoning, the solution is

$$x = x_0 \cos\left(\sqrt{\frac{k}{m}}t\right). \tag{8.5.25}$$

The graph of x for $k = 10$, $m = 5$, and $x_0 = 2$ is shown in Figure 8.5.4.

Notice that the period of the motion is

$$T = \frac{2\pi}{\sqrt{\frac{k}{m}}} = 2\pi\sqrt{\frac{m}{k}}. \tag{8.5.26}$$

The frequency of the motion, that is, the number of complete oscillations in one unit of time, is

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}. \tag{8.5.27}$$

Hence for a fixed mass, increasing the spring constant, that is, increasing the stiffness of the spring, decreases the period and increases the frequency; for a fixed spring constant, increasing the mass increases the period and decreases the frequency.

Now suppose there is a damping force, a force resisting the motion of the object, which is proportional to the velocity. This adds an additional term of $-c\dot{x}$, where c is a positive constant, to the force acting on the object, giving us $F = -kx - c\dot{x}$. Thus

$$m\ddot{x} = -kx - c\dot{x}, \tag{8.5.28}$$

and so

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0 \tag{8.5.29}$$

replaces (8.5.24) as the equation describing the motion of the object. To simplify the notation, we will let

$$b = \frac{c}{2m}$$

and

$$a = \sqrt{\frac{k}{m}}.$$

Then our differential equation becomes

$$\ddot{x} + 2b\dot{x} + a^2 x = 0, \tag{8.5.30}$$

with characteristic equation (using s for the variable)

$$s^2 + 2bs + a^2 = 0. \tag{8.5.31}$$

Hence the roots of the characteristic equation are

$$s_1 = \frac{-2b - \sqrt{4b^2 - 4a^2}}{2} = -b - \sqrt{b^2 - a^2} \tag{8.5.32}$$

and

$$s_2 = \frac{-2b + \sqrt{4b^2 - 4a^2}}{2} = -b + \sqrt{b^2 - a^2}. \tag{8.5.33}$$

Thus the behavior of the system depends on whether $b^2 - a^2 > 0$, $b^2 - a^2 = 0$, or $b^2 - a^2 < 0$. Equivalently, since

$$b^2 - a^2 = \frac{c^2}{4m^2} - \frac{k}{m},$$

the behavior of the system depends on whether $c^2 > 4mk$, $c^2 = 4mk$, or $c^2 < 4mk$. In the first case the system is said to be *overdamped*, in the second it is *critically damped*, and in the third it is *underdamped*.

First consider the overdamped case $b^2 - a^2 > 0$. In this case the characteristic equation has distinct real roots, so the general solution is

$$x = c_1 e^{s_1 t} + c_2 e^{s_2 t}. \tag{8.5.34}$$

Now

$$\dot{x} = c_1 s_1 e^{s_1 t} + c_2 s_2 e^{s_2 t}, \tag{8.5.35}$$

so $x(0) = c_1 + c_2$ and $\dot{x}(0) = c_1 s_1 + c_2 s_2$. Hence the initial conditions, $x(0) = x_0$ and $\dot{x}(0) = 0$, give us

$$x_0 = c_1 + c_2$$

and

$$0 = c_1 s_1 + c_2 s_2.$$

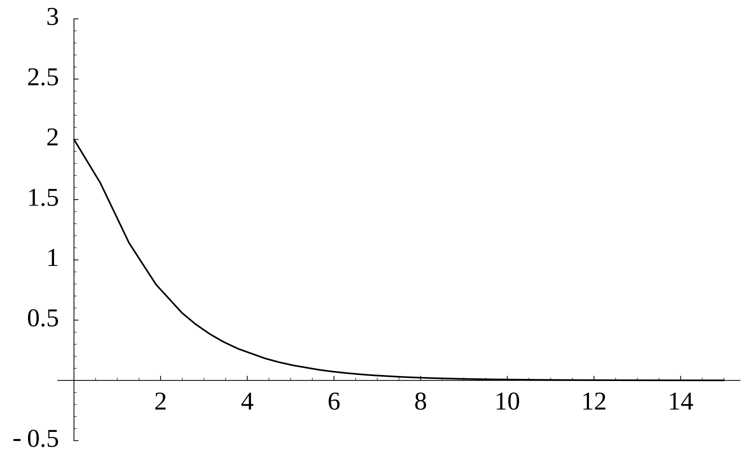

Figure 8.5.5 Motion of an overdamped mass-spring system

Multiplying the first equation by s_1 and subtracting from the second gives us

$$-x_0 s_1 = c_2(s_2 - s_1).$$

Hence

$$c_2 = -\frac{x_0 s_1}{s_2 - s_1}$$

and

$$c_1 = x_0 - c_2 = \frac{x_0(s_2 - s_1)}{s_2 - s_1} + \frac{x_0 s_1}{s_2 - s_1} = \frac{x_0 s_2}{s_2 - s_1}.$$

Thus

$$x = \frac{x_0}{s_2 - s_1}(s_2 e^{s_1 t} - s_1 e^{s_2 t}). \tag{8.5.36}$$

Now $b > 0$ and $b > \sqrt{b^2 - a^2}$, so

$$s_2 = -b + \sqrt{b^2 - a^2} < 0.$$

Hence

$$s_1 < s_2 < 0. \tag{8.5.37}$$

It follows that $e^{s_2 t} > e^{s_1 t}$, $s_2 - s_1 > 0$, and

$$s_2 e^{s_1 t} - s_1 e^{s_2 t} > s_2 e^{s_2 t} - s_1 e^{s_2 t} = e^{s_2 t}(s_2 - s_1) > 0$$

for all $t \geq 0$. Hence if $x_0 < 0$, then $x(t) < 0$ for all $t \geq 0$, and if $x_0 > 0$, then $x(t) > 0$ for all $t > 0$. Combining this with

$$\lim_{t\to\infty} x(t) = 0, \tag{8.5.38}$$

we see that in this case the system does not oscillate at all. After release, the object simply returns to the equilibrium position. Figure 8.5.5 shows this behavior for $k = 10$, $m = 5$, $c = 20$, and $x_0 = 2$.

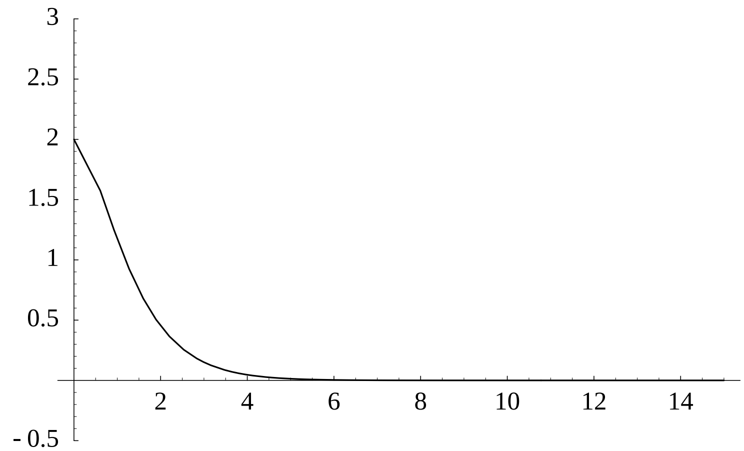

Figure 8.5.6 Motion of a critically damped mass-spring system

Next consider the case when $b^2 - a^2 = 0$. In this case the characteristic equation has only one real root, $s_1 = s_2 = -b$, so the general solution is

$$x = c_1 e^{-bt} + c_2 t e^{-bt}. \tag{8.5.39}$$

Then

$$\dot{x} = -bc_1 e^{-bt} - bc_2 t e^{-bt} + c_2 e^{-bt}, \tag{8.5.40}$$

so $x(0) = c_1$ and $\dot{x}(0) = -bc_1 + c_2$. Hence the initial conditions, $x(0) = x_0$ and $\dot{x}(0) = 0$, give us $c_1 = x_0$ and $c_2 = bx_0$. Thus

$$x = x_0 e^{-bt} + bx_0 t e^{-bt} = x_0 e^{-bt}(1 + bt). \tag{8.5.41}$$

Equivalently, since $b = \frac{c}{2m}$,

$$x = x_0 e^{-\frac{c}{2m}t}\left(1 + \frac{c}{2m}t\right). \tag{8.5.42}$$

Now for any $t \geq 0$,

$$1 + \frac{c}{2m}t > 0.$$

Hence, as in the overdamped case, the system does not oscillate. Once released, the object moves back to the equilibrium position without ever crossing it. Figure 8.5.6 shows this behavior for $k = 10$, $m = 5$, $c = 10\sqrt{2}$, and $x_0 = 2$. This motion is said to be critically damped because any increase in c results in overdamped motion, while any decrease in c results in underdamped motion, which we consider next.

Finally, consider the case when $b^2 - a^2 < 0$. The roots of the characteristic equation are now

$$s_1 = -b - \sqrt{b^2 - a^2} = -b - i\sqrt{a^2 - b^2} \tag{8.5.43}$$

and

$$s_2 = -b + \sqrt{b^2 - a^2} = -b + i\sqrt{a^2 - b^2} \tag{8.5.44}$$

If we let $\alpha = \sqrt{a^2 - b^2}$, then the general solution is

$$x = e^{-bt}(c_1 \cos(\alpha t) + c_2 \sin(\alpha t)). \tag{8.5.45}$$

Then

$$\dot{x} = e^{-bt}(-\alpha c_1 \sin(\alpha t) + \alpha c_2 \cos(\alpha t)) - be^{-bt}(c_1 \cos(\alpha t) + c_2 \sin(\alpha t)), \tag{8.5.46}$$

so $x(0) = c_1$ and $\dot{x}(0) = \alpha c_2 - bc_1$. Hence the initial conditions, $x(0) = x_0$ and $\dot{x}(0) = 0$, imply that $c_1 = x_0$ and

$$c_2 = \frac{bx_0}{\alpha}.$$

Thus

$$x = e^{-bt}\left(x_0 \cos(\alpha t) + \frac{bx_0}{\alpha} \sin(\alpha t)\right) = \frac{x_0}{\alpha} e^{-bt}(\alpha \cos(\alpha t) + b \sin(\alpha t)). \tag{8.5.47}$$

This expression simplifies somewhat if we introduce the angle

$$\theta = \tan^{-1}\left(\frac{b}{\alpha}\right). \tag{8.5.48}$$

Then

$$\cos(\theta) = \frac{\alpha}{\sqrt{\alpha^2 + b^2}}$$

and

$$\sin(\theta) = \frac{b}{\sqrt{\alpha^2 + b^2}}.$$

Moreover, since $\alpha = \sqrt{a^2 - b^2}$,

$$\sqrt{\alpha^2 + b^2} = \sqrt{(a^2 - b^2) + b^2} = a = \sqrt{\frac{k}{m}}.$$

Hence

$$\begin{aligned} x &= \frac{x_0\sqrt{\alpha^2 + b^2}}{\alpha} e^{-bt}\left(\frac{\alpha}{\sqrt{\alpha^2 + b^2}} \cos(\alpha t) + \frac{b}{\sqrt{\alpha^2 + b^2}} \sin(\alpha t)\right) \\ &= \frac{x_0}{\alpha}\sqrt{\frac{k}{m}} e^{-bt}(\cos(\theta)\cos(\alpha t) + \sin(\theta)\sin(\alpha t)). \end{aligned}$$

Using the angle subtraction formula for cosine, this becomes

$$x = \frac{x_0}{\alpha}\sqrt{\frac{k}{m}} e^{-bt} \cos(\alpha t - \theta). \tag{8.5.49}$$

The presence of the cosine factor in this expression shows us that, even though we still have

$$\lim_{t \to \infty} x(t) = 0,$$

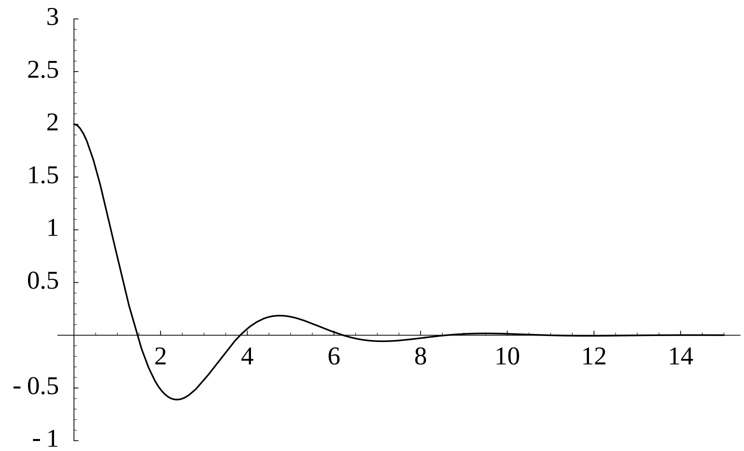

Figure 8.5.7 Motion of an underdamped mass-spring system

the underdamped mass-spring system will oscillate about the equilibrium position with a decreasing amplitude of

$$\frac{x_0}{\alpha}\sqrt{\frac{k}{m}}e^{-bt}. \tag{8.5.50}$$

Figure 8.5.7 shows this behavior for $k = 10$, $m = 5$, $c = 5$, and $x_0 = 2$.

Problems

1. In an experiment to determine g, a pendulum of length 50 centimeters is observed to have a period of oscillation of 1.42 seconds. Approximate g based on this observation.

2. The period of oscillation of a pendulum of length b given in (8.5.19) is, as mentioned, only an approximation of the true period. It can be shown that the true period of a pendulum released from an angle α is given by

$$T = 4\sqrt{\frac{b}{g}}\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - k^2\sin^2(\phi)}}\,d\phi,$$

   where $0 < \alpha < \pi$ and $k = \sin\left(\frac{\alpha}{2}\right)$.

   (a) Find the period of oscillation for a pendulum of length 50 centimeters for $\alpha = \frac{\pi}{4}$, $\alpha = \frac{\pi}{6}$, $\alpha = \frac{\pi}{50}$, and $\alpha = \frac{\pi}{100}$. Compare these results with the approximation given in (8.5.19).

   (b) Graph T as a function of α for $-\frac{\pi}{4} \leq \alpha \leq \frac{\pi}{4}$. For comparison, also plot the horizontal line

$$T = 2\pi\sqrt{\frac{b}{g}}.$$

3. Consider a mass-spring system with $x_0 = 10$, $\dot{x}(0) = 0$, $k = 10$, and $m = 10$. Plot $x(t)$ for $c = 0$, $c = 5$, $c = 10$, $c = 20$, $c = 25$, and $c = 30$. Identify each motion as overdamped, critically damped, underdamped, or undamped.

4. Consider a mass-spring system with $x_0 = 10$, $\dot{x}(0) = 0$, $m = 10$, and $c = 20$. Plot $x(t)$ for $k = 2$, $k = 5$, $k = 10$, and $k = 15$. Identify each motion as overdamped, critically damped, underdamped, or undamped.

5. Consider the underdamped motion of a mass-spring system expressed in (8.5.49).

   (a) Show that the maximum values of $x(t)$ occur at $t = 0, T, 2T, \ldots$, where

   $$T = \frac{2\pi}{\sqrt{\dfrac{k}{m} - \dfrac{c^2}{4m^2}}}.$$

   Note that when $c = 0$, T reduces to the period of the motion for the mass-spring system without damping.

   (b) Show that if x_1 and x_2 are two successive maximum values of $x(t)$, then

   $$\frac{x_1}{x_2} = e^{\frac{cT}{2m}}.$$

6. Inside the earth, the force of gravity acting on an object is proportional to the distance between the object and the center of the earth.

   (a) Suppose a hole is drilled through the earth from pole to pole and a rock is dropped into the hole. If $x(t)$ is the distance from the object to the center of the earth at time t, show that, ignoring any resistive forces,

   $$x = R\cos\left(\sqrt{\frac{g}{R}}t\right),$$

   where R is the radius of the earth.

   (b) How long, in minutes, does it take for the rock to make one complete trip from pole to pole and back? Use $R = 3950$ miles.

   (c) What is the velocity of the rock, in miles per hour, when it reaches the center of the earth?