

***Difference Equations
to
Differential Equations***

**Section 8.4
Second Order Linear
Differential Equations**

To this point we have we have considered only first order differential equations. However, many of the most interesting differential equations involve second derivatives. Indeed, since acceleration is the second derivative of position, Newton's second law of motion, $F = ma$, is a second order differential equation. In general, if f is a known function of three variables, then the equation

$$\ddot{x} = f(\dot{x}, x, t) \tag{8.4.1}$$

is called a *second order differential equation*. If we let $y = \dot{x}$, then (8.4.1) may be written as a pair of first order differential equations

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= f(y, x, t). \end{aligned} \tag{8.4.2}$$

Hence moving from the study of first order differential equations to the study of second order differential equations is analogous to moving from the study of one algebraic equation in one unknown to the study of two algebraic equations in two unknowns. We will make use of this fact when we consider numerical approximations to solutions of second order equations in Section 8.6.

As was the case with first order equations, the existence of a closed form solution to a second order differential equation and our ability to find one when it exists depends very much on the form of the function f in (8.4.1). We shall consider closed form solutions for only one class of such equations, leaving other equations for either the numerical approximations of Section 8.6 or the infinite series techniques of Section 8.7. Here we are concerned with equations of the form

$$\ddot{x} + b\dot{x} + cx = 0, \tag{8.4.3}$$

which we call a *second order homogeneous linear differential equation with constant coefficients*, corresponding to

$$f(\dot{x}, x, t) = -b\dot{x} - cx$$

in (8.4.1). The term *homogeneous* refers to the fact that the function $x(t) = 0$ for all t is a solution of the equation and the phrase *constant coefficients* refers to the fact that b and c are assumed to be constants.

To begin our study of these equations, suppose $x_1(t)$ and $x_2(t)$ are both solutions of (8.4.3) and let $x(t) = c_1x_1(t) + c_2x_2(t)$ for constants c_1 and c_2 . Then

$$\begin{aligned}
 \ddot{x} + b\dot{x} + cx &= (c_1\ddot{x}_1 + c_2\ddot{x}_2) + b(c_1\dot{x}_1 + c_2\dot{x}_2) + c(c_1x_1 + c_2x_2) \\
 &= c_1(\ddot{x}_1 + b\dot{x}_1 + cx_1) + c_2(\ddot{x}_2 + b\dot{x}_2 + cx_2) \\
 &= (c_1)(0) + (c_2)(0) = 0.
 \end{aligned}$$

That is, x is also a solution of (8.4.3).

Proposition If x_1 and x_2 are both solutions of the equation

$$\ddot{x} + b\dot{x} + cx = 0,$$

then $x = c_1x_1 + c_2x_2$ is also a solution of this equation for any constants c_1 and c_2 .

The next proposition is key to our method of solving equations of the form (8.4.3), although we will leave its justification to a more advanced course. First we introduce a definition which will make the proposition, as well as our later results, easier to state.

Definition If f and g are functions for which neither one is a constant multiple of the other, then we say f and g are *linearly independent*.

Proposition Suppose x_1 and x_2 are linearly independent solutions of the equation

$$\ddot{x} + b\dot{x} + cx = 0.$$

Then for any solution x , there exist constants c_1 and c_2 such that

$$x = c_1x_1 + c_2x_2. \tag{8.4.4}$$

The equation

$$\ddot{x} + b\dot{x} + cx = 0 \tag{8.4.5}$$

will have a unique solution only when we place some restrictions on x . For example, if we specify initial conditions for both x and \dot{x} , say, $x(t_0) = x_0$ and $\dot{x}(t_0) = y_0$, then (8.4.4) will have a unique solution which satisfies these conditions. This statement is far from obvious, but should appear reasonable in light of the observation, made above, that we could write this equation as a pair of first order equations

$$\begin{aligned}
 \dot{x} &= y \\
 \dot{y} &= -by - cx.
 \end{aligned} \tag{8.4.6}$$

Hence our method of attack in solving (8.4.4) will be to first find two linearly independent solutions, say, x_1 and x_2 , and then find values for constants c_1 and c_2 such that $x = c_1x_1 + c_2x_2$ satisfies the given initial conditions.

To find two linearly independent solutions of (8.4.4), we begin with the observation that if x satisfies this equation, then \ddot{x} is equal to a sum of constant multiples of x and \dot{x} . Hence it would be reasonable to begin with $x = e^{kt}$, for some constant k , as an initial guess. In that case,

$$\dot{x} = ke^{kt}$$

and

$$\ddot{x} = k^2 e^{kt},$$

so x will be a solution of (8.4.4) if and only if

$$k^2 e^{kt} + bke^{kt} + ce^{kt} = e^{kt}(k^2 + bk + c) = 0 \quad (8.4.7)$$

for all t . Since $e^{kt} \neq 0$ for all t , this will happen if and only if

$$k^2 + bk + c = 0. \quad (8.4.8)$$

Hence $x = e^{kt}$ is a solution to (8.4.4) if and only if k is a root of (8.4.8).

Definition The equation

$$k^2 + bk + c = 0 \quad (8.4.9)$$

is called the *characteristic equation* of the differential equation

$$\ddot{x} + b\dot{x} + cx = 0.$$

Since the characteristic equation is quadratic in k , its roots are given by the quadratic formula, namely,

$$k_1 = \frac{-b - \sqrt{b^2 - 4c}}{2} \quad (8.4.10)$$

and

$$k_2 = \frac{-b + \sqrt{b^2 - 4c}}{2}. \quad (8.4.11)$$

At this point, our search for solutions breaks into three cases, depending on whether k_1 and k_2 are (1) distinct real numbers (that is, $b^2 - 4c > 0$), (2) distinct complex numbers (that is, $b^2 - 4c < 0$), or (3) real, but equal (that is, $b^2 - 4c = 0$).

Case 1: Distinct real roots

Suppose k_1 and k_2 are distinct real roots of the characteristic equation. In that case, $x_1 = e^{k_1 t}$ and $x_2 = e^{k_2 t}$ are linearly independent solutions of

$$\ddot{x} + b\dot{x} + cx = 0$$

and all that remains is to find constants c_1 and c_2 such that

$$x = c_1 e^{k_1 t} + c_2 e^{k_2 t} \quad (8.4.12)$$

satisfies the given initial conditions.

Example Consider the equation

$$\ddot{x} + \dot{x} - 6x = 0$$

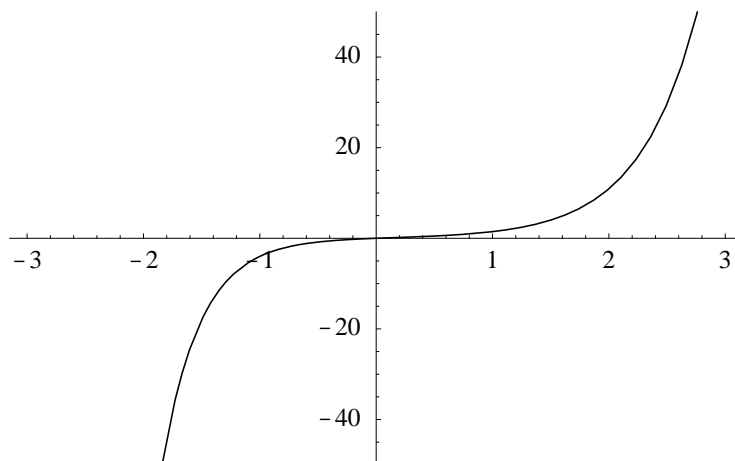


Figure 8.4.1 Solution of $\ddot{x} + \dot{x} - 6x = 0$ with $x(0) = 0$ and $\dot{x}(0) = 1$

with initial conditions $x(0) = 0$ and $\dot{x}(0) = 1$. For the characteristic equation we have

$$0 = k^2 + k - 6 = (k + 3)(k - 2).$$

Hence the roots of the characteristic equation are $k_1 = -3$ and $k_2 = 2$. Thus we must have

$$x = c_1 e^{-3t} + c_2 e^{2t}$$

for some constants c_1 and c_2 . Now

$$\dot{x} = -3c_1 e^{-3t} + 2c_2 e^{2t},$$

so

$$x(0) = c_1 + c_2$$

and

$$\dot{x}(0) = -3c_1 + 2c_2.$$

Hence the initial conditions imply that

$$c_1 + c_2 = 0$$

$$-3c_1 + 2c_2 = 1.$$

The first equation implies $c_1 = -c_2$. Substituting into the second equation, we have

$$1 = 3c_2 + 2c_2 = 5c_2.$$

Hence

$$c_2 = \frac{1}{5}$$

and

$$c_1 = -\frac{1}{5}.$$

Thus

$$x = -\frac{1}{5}e^{-3t} + \frac{1}{5}e^{2t}.$$

The graph of x is shown in Figure 8.4.1

Case 2: Complex roots

Suppose k_1 and k_2 are distinct complex roots of the characteristic equation. As before, $e^{k_1 t}$ and $e^{k_2 t}$ are linearly independent solutions of

$$\ddot{x} + b\dot{x} + cx = 0.$$

However, these are complex-valued functions and for most applications we are looking for real-valued solutions. Now if we let

$$p = -\frac{b}{2} \tag{8.4.13}$$

and

$$q = \frac{\sqrt{4c - b^2}}{2}, \tag{8.4.14}$$

then $k_1 = p - qi$ and $k_2 = p + qi$. Hence

$$e^{k_1 t} = e^{(p-qi)t} = e^{pt} e^{-iqt} = e^{pt} (\cos(qt) - i \sin(qt)) \tag{8.4.15}$$

and

$$e^{k_2 t} = e^{(p+qi)t} = e^{pt} e^{iqt} = e^{pt} (\cos(qt) + i \sin(qt)). \tag{8.4.16}$$

Since these are both solutions, we know that

$$x_1 = \frac{1}{2} e^{k_1 t} + \frac{1}{2} e^{k_2 t} = e^{pt} \cos(qt) \tag{8.4.17}$$

and

$$x_2 = \frac{1}{2i} e^{k_2 t} - \frac{1}{2i} e^{k_1 t} = e^{pt} \sin(qt) \tag{8.4.18}$$

are also solutions. Then x_1 and x_2 are linearly independent real-valued solutions, so any real-valued solution must be of the form

$$x = c_1 x_1 + c_2 x_2 = e^{pt} (c_1 \cos(qt) + c_2 \sin(qt)) \tag{8.4.19}$$

for some constants c_1 and c_2 .

Example Consider the equation

$$\ddot{x} + 2\dot{x} + 5x = 0$$

with initial conditions $x(0) = 2$ and $\dot{x}(0) = 0$. The characteristic equation is

$$k^2 + 2k + 5 = 0,$$

which has roots

$$\frac{-2 \pm \sqrt{4 - 20}}{2} = -1 \pm 2i.$$

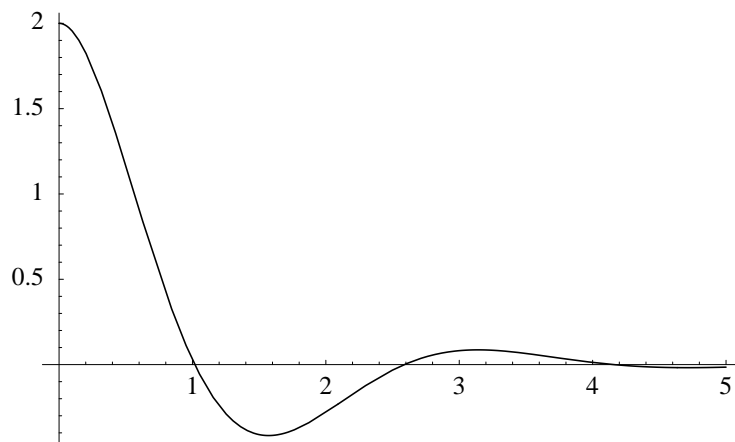


Figure 8.4.2 Solution of $\ddot{x} + 2\dot{x} + 5x = 0$ with $x(0) = 2$ and $\dot{x}(0) = 0$

Hence, by (8.4.19), we must have

$$x = e^{-t}(c_1 \cos(2t) + c_2 \sin(2t))$$

for some constants c_1 and c_2 . Now

$$\dot{x} = e^{-t}(-2c_1 \sin(2t) + 2c_2 \cos(2t)) - e^{-t}(c_1 \cos(2t) + c_2 \sin(2t)),$$

so

$$x(0) = c_1$$

and

$$\dot{x}(0) = 2c_2 - c_1.$$

Hence the initial conditions $x(0) = 2$ and $\dot{x}(0) = 0$ imply that $c_1 = 2$ and

$$0 = 2c_2 - 2.$$

Thus $c_2 = 1$ and we have

$$x = e^{-t}(2 \cos(2t) + \sin(2t)).$$

The graph of x is shown in Figure 8.4.2.

Case 3: Single real root

Suppose the characteristic equation has a single real root. In this case,

$$k_1 = k_2 = -\frac{b}{2}. \quad (8.4.20)$$

For simplicity, let us call this common value k . Then $x_1 = e^{kt}$ is a solution of the equation

$$\ddot{x} + b\dot{x} + cx = 0,$$

but in order to specify all possible solutions we need to find another solution which is linearly independent of x_1 . We will show that, in this case, $x_2 = te^{kt}$ is such a solution. Now

$$\dot{x}_2 = kte^{kt} + e^{kt} = (1 + kt)e^{kt} \quad (8.4.21)$$

and

$$\ddot{x}_2 = (1 + kt)ke^{kt} + ke^{kt} = (2k + k^2t)e^{kt}. \quad (8.4.22)$$

Hence, remembering that k is a root of the characteristic equation (that is, $k^2 + bk + c = 0$) and $k = -\frac{b}{2}$, we have

$$\begin{aligned} \ddot{x}_2 + b\dot{x}_2 + cx_2 &= (2k + k^2t)e^{kt} + b(1 + kt)e^{kt} + cte^{kt} \\ &= e^{kt}(2k + k^2t + b + bkt + ct) \\ &= e^{kt}((k^2 + bk + c)t + 2k + b) \\ &= e^{kt}(2k + b) \\ &= e^{kt}\left(-\frac{2b}{2} + b\right) \\ &= 0 \end{aligned}$$

for all t . Hence x_2 is another solution, clearly linearly independent of x_1 . Thus for any solution x , there exist constants c_1 and c_2 such that

$$x = c_1x_1 + c_2x_2 = c_1e^{kt} + c_2te^{kt}. \quad (8.4.23)$$

Example Consider the equation

$$\ddot{x} + 2\dot{x} + x = 0$$

with initial conditions $x(0) = 10$ and $\dot{x}(0) = -20$. The characteristic equation is

$$0 = k^2 + 2k + 1 = (k + 1)^2$$

which has the single root $k = -1$. Hence

$$x = c_1e^{-t} + c_2te^{-t}$$

for some constants c_1 and c_2 . Now

$$\dot{x} = -c_1e^{-t} - c_2te^{-t} + c_2e^{-t},$$

so $x(0) = c_1$ and $\dot{x}(0) = -c_1 + c_2$. Hence the initial conditions $x(0) = 10$ and $\dot{x}(0) = -20$ imply that $c_1 = 10$ and

$$-20 = -10 + c_2.$$

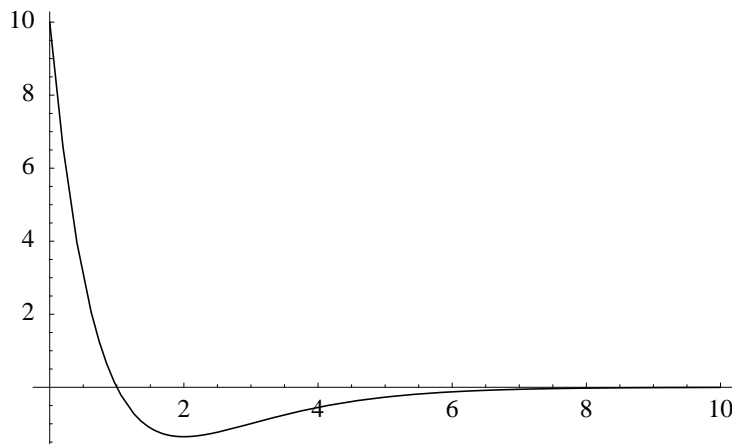


Figure 8.4.3 Solution of $\ddot{x} + 2\dot{x} + x = 0$ with $x(0) = 10$ and $\dot{x}(0) = -20$

Thus $c_2 = -10$ and we have

$$x = 10e^{-t} - 10te^{-t} = 10(1 - t)e^{-t}.$$

The graph of x is shown in Figure 8.4.3.

Summary

If x_1 and x_2 are linearly independent solutions of the equation

$$\ddot{x} + b\dot{x} + cx = 0, \quad (8.4.24)$$

then any solution of (8.4.24) is of the form $x = c_1x_1 + c_2x_2$ for some constants c_1 and c_2 . The family of all solutions $x = c_1x_1 + c_2x_2$ is called the *general solution* of (8.4.24). A solution with specified values for c_1 and c_2 is called a *particular solution*.

Let k_1 and k_2 be the roots of the characteristic equation

$$k^2 + bk + c = 0. \quad (8.4.25)$$

If k_1 and k_2 are real numbers with $k_1 \neq k_2$, then the general solution of (8.4.24) is

$$x = c_1e^{k_1t} + c_2e^{k_2t}. \quad (8.4.26)$$

If k_1 and k_2 are complex numbers with $k_1 = p - qi$ and $k_2 = p + qi$, then the general solution of (8.4.24) is

$$x = e^{pt}(c_1 \cos(qt) + c_2 \sin(qt)). \quad (8.4.27)$$

Finally, if $k = k_1 = k_2$, then the general solution of (8.4.24) is

$$x = c_1e^{kt} + c_2te^{kt}. \quad (8.4.28)$$

In the next section we will discuss the motion of a pendulum and the motion of a mass vibrating at the end of a spring as applications of the equations considered in this section.

Problems

1. Solve each of the following differential equations and plot the solution.
 - (a) $\ddot{x} + \dot{x} - 2x = 0$, $x(0) = 0$, $\dot{x}(0) = 2$
 - (b) $\ddot{x} = -x$, $x(0) = 10$, $\dot{x}(0) = 5$
 - (c) $\ddot{x} + 3\dot{x} + 2x = 0$, $x(0) = 1$, $\dot{x}(0) = 0$
 - (d) $\ddot{x} - 4\dot{x} + 4x = 0$, $x(0) = 5$, $\dot{x}(0) = 0$
 - (e) $\ddot{x} - 2\dot{x} + 2x = 0$, $x(0) = 10$, $\dot{x}(0) = 4$
 - (f) $-\ddot{x} + 2\dot{x} - 4x = 0$, $x(0) = 1$, $\dot{x}(0) = 0$
 - (g) $\ddot{x} + 4\dot{x} + 20x = 0$, $x(0) = 0$, $\dot{x}(0) = 3$
 - (h) $2\ddot{x} + 3\dot{x} - 2x = 0$, $x(0) = 0$, $\dot{x}(0) = -2$
 - (i) $\ddot{x} + 6\dot{x} + 9x = 0$, $x(0) = -6$, $\dot{x}(0) = 4$
2. Consider the equation $\ddot{x} + 2\dot{x} - 3x = 0$.
 - (a) If $\dot{x}(0) = 1$, plot the solutions for $x(0) = 0$, $x(0) = -5$, and $x(0) = 5$. How do these solutions compare?
 - (b) If $x(0) = 0$, plot the solutions for $\dot{x}(0) = 0$, $\dot{x}(0) = -2$, and $\dot{x}(0) = 2$. How do these solutions compare?
3. Consider the equation $\ddot{x} + 2\dot{x} + 10x = 0$.
 - (a) If $\dot{x}(0) = 1$, plot the solutions for $x(0) = 0$, $x(0) = -10$, and $x(0) = 10$. How do these solutions compare?
 - (b) If $x(0) = 10$, plot the solutions for $\dot{x}(0) = 0$, $\dot{x}(0) = -5$, and $\dot{x}(0) = 5$. How do these solutions compare?
4. Consider the equation $\ddot{x} + 4\dot{x} + 4x = 0$.
 - (a) If $\dot{x}(0) = -15$, plot the solutions for $x(0) = 0$, $x(0) = -5$, and $x(0) = 5$. How do these solutions compare?
 - (b) If $x(0) = 10$, plot the solutions for $\dot{x}(0) = 0$, $\dot{x}(0) = -20$, and $\dot{x}(0) = 20$. How do these solutions compare?
5. The techniques developed in this section may be used to solve higher order homogeneous linear differential equations with constant coefficients. Generalize the methods of this section to find the general solution for each of the following equations.
 - (a) $\frac{d^3x}{dt^3} + 2\frac{d^2x}{dt^2} - \frac{dx}{dt} - 2x = 0$
 - (b) $\frac{d^3x}{dt^3} + 3\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + x = 0$
6. Show that if b and c are both positive and x is a solution of

$$\ddot{x} + b\dot{x} + cx = 0,$$

then $\lim_{t \rightarrow \infty} x(t) = 0$.