Difference Equations *to* Differential Equations	**Section 8.1** **Numerical Solutions of Differential Equations**

If x is a function of a real variable t and f is a function of both x and t, then the equation

$$\dot{x}(t) = f(x(t), t) \qquad (8.1.1)$$

is called a *first order differential equation*. Solving such an equation involves more than algebraic manipulation; indeed, although the equation itself involves three quantities, x, $\dot{x}$, and t, to find a solution we must identify a function x, defined solely in terms of the independent variable t, which satisfies the relationship of (8.1.1) for all t in some open interval. For many equations, exact solution is not possible and we have to rely on approximations. In this chapter we will discuss techniques for finding both approximate and, where possible, exact solutions to differential equations.

We have already seen many examples of differential equations: in Section 4.8 when we discussed finding the position of an object moving in a straight line given its velocity function and its initial position, in Section 6.3 when we discussed models for growth and decay, in Section 7.3 when we discussed the motion of a projectile, and in Section 7.4 when we considered the two-body problem. Indeed, in many ways the study of differential equations is at the heart of calculus. To study the interaction of physical bodies in the world is to study the ramifications of physical laws such as the law of gravitation and Newton's second law of motion, laws which frequently lead, as we saw in Section 7.4, to questions involving the solution of differential equations. Newton was the first to realize the power of calculus for solving a vast array of physical problems. The mathematicians that followed him enlarged and refined his techniques until they began to believe that the entire future of the universe, as well as its past, could be discerned from a knowledge of the current positions and velocities of all physical bodies and the forces at work between them. In such a world view, nothing is undetermined in itself; what appears to us as undetermined is simply a reflection of our ignorance of the forces involved. As an example of this view, writing in 1795, Pierre Simon Laplace (1749-1827) said:

> Given for one instant an intelligence which could comprehend all the forces by which nature is animated and the respective situation of the beings who compose it — an intelligence sufficiently vast to submit these data to analysis — it would embrace in the same formula the movements of the greater bodies of the universe and those of the lightest atom; for it, nothing would be uncertain and the future, as the past, would be present to its eyes. *

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* P. S. Laplace, *Essai philosphique sur les probabilités* (Paris, 1814), translated by F. W. Truscott and F. L. Emory, *A Philosophical Essay on Probabilities* (New York, 1951), page 3.

Today we know more about the limits to our knowledge, but, nevertheless, the study of differential equations remains a key component to our understanding of the universe.

To begin our study, consider the equation

$$\dot{x}(t) = f(x(t), t) \tag{8.1.2}$$

with the initial condition $x(t_0) = x_0$. To simplify notation, we will frequently omit the independent variable when referring to $x(t)$ and write simply

$$\dot{x} = f(x, t). \tag{8.1.3}$$

Now if $f(x, t)$ depends only on the value of t, that is, if $f(x, t) = g(t)$ for all values of x, where g is a function of t alone, then we may solve (8.1.3) by integration. That is, integrating both sides (8.1.3) gives us

$$\int_{t_0}^{t} \dot{x}(s) ds = \int_{t_0}^{t} g(s) ds. \tag{8.1.4}$$

Substituting

$$\int_{t_0}^{t} \dot{x}(s) ds = x(t) - x(t_0),$$

into (8.1.4), we have

$$x(t) = x(t_0) + \int_{t_0}^{t} g(s) ds. \tag{8.1.5}$$

Assuming we can compute the definite integral, which, provided g is continuous, can at least always be done numerically, we have solved the differential equation. This is the type of differential equation we considered in Section 4.8.

Example   Consider the equation

$$\dot{x} = 4 \sin(3t)$$

with initial condition $x(0) = 5$. Then

$$\begin{aligned}
x(t) &= 5 + \int_{0}^{t} 4 \sin(3s) ds \\
&= 5 - \frac{4}{3} \cos(3s) \Big|_{0}^{t} \\
&= 5 - \frac{4}{3} \cos(3t) + \frac{4}{3} \\
&= \frac{1}{3}(19 - 4 \cos(3t)).
\end{aligned}$$

More generally, suppose f in (8.1.3) depends on both x and t. In that case, since the right-hand side of the equation involves the unknown function x, we cannot simply

integrate both sides of the equation. We have solved some equations like this in earlier sections, such as the inhibited population growth model

$$\dot{x} = \frac{\alpha}{M} x(M - x)$$

in Section 6.3, and we will discuss general techniques for finding exact solutions for several types of equations in the coming sections. However, in this section we will concentrate on direct numerical techniques for approximating solutions. Indeed, knowing the difficulty of evaluating ordinary definite integrals, it is not hard to believe that many, if not most, differential equations may be solved only through numerical approximation.

Although the function x in equation (8.1.3) is unknown, we do have enough information to find its best affine approximation at t_0. Namely, the best affine approximation to x at t_0 is

$$T(t) = x_0 + \dot{x}(t_0)(t - t_0) = x_0 + f(x_0, t_0)(t - t_0). \tag{8.1.6}$$

Hence, from our work with Taylor's theorem in Section 5.2 and assuming $f(x(t), t)$ is a continuous function, we have

$$x(t) = x_0 + f(x_0, t_0)(t - t_0) + O((t - t_0)^2). \tag{8.1.7}$$

Equivalently,
$$x(t_0 + h) = x_0 + hf(x_0, t_0) + O(h^2). \tag{8.1.8}$$

Thus for a small value of h,
$$x_1 = x_0 + hf(x_0, t_0)$$

will provide a good approximation to $x(t_0 + h)$. However, we want to do more than this; since x is a function, we want to be able to approximate its values over an entire interval, say $[t_0, t_1]$. To do so, we choose a small value for h and iterate the process that gave us x_1. Specifically, we let $s_k = t_0 + kh$, $k = 0, 1, \ldots, n$, where n is chosen large enough that $s_n \geq t_1$, and compute

$$x_{k+1} = x_k + hf(x_k, s_k) \tag{8.1.9}$$

for $k = 0, 1, \ldots, n - 1$. That is, we compute an approximation for $x(s_{k+1})$ by applying the best affine approximation to our approximation for $x(s_k)$, repeating the process enough times until we have approximated x over the entire interval $[t_0, t_1]$. This method of approximation is known as *Euler's method.*

Euler's method   To approximate a solution to the equation

$$\dot{x} = f(x, t)$$

with initial condition $x(t_0) = x_0$ on an interval $[t_0, t_1]$, choose a small value for $h > 0$ and an integer n such that $t_0 + nh \geq t_1$. Letting $s_k = t_0 + kh$, $k = 0, 1, \ldots, n$, compute $x_1, x_2, \ldots, x_n$ using the difference equation

$$x_{k+1} = x_k + hf(x_k, s_k). \tag{8.1.10}$$

Then x_k is an approximation for $x(t + kh)$.

Note that (8.1.10) makes use of a difference equation, a discrete time equation that we first met in Section 1.4, in order to approximate the solution of a differential equation.

Example   Consider the differential equation $\dot{x} = 0.04x$ with initial condition $x(0) = 50$. From our work in Chapter 6 we know that the solution to this equation is

$$x(t) = 50e^{0.04t}.$$

In particular, $x(50) = 50e^2 = 369.45$, rounding our answer to the second decimal place. To approximate x on the interval $[0, 50]$ using Euler's method, we will first take $h = 1$ and, starting with $x_0 = 50$, compute $x_1, x_2, \ldots, x_{50}$, where in this case x_k will approximate $x(k)$. Using (8.1.10) with $f(x, t) = 0.04x$ and rounding to two decimal places, we have

$$x_1 = x_0 + h(0.04x_0) = 50 + (1)(0.04)(50) = 50 + 2 + 52.00,$$
$$x_2 = x_1 + h(0.04x_1) = 52 + (1)(0.04)(52) = 52 + 2.08 = 54.08,$$
$$x_3 = x_2 + h(0.04x_2) = 54.08 + (1)(0.04)(54.08) = 54.08 + 2.16 = 56.24,$$

and so on, ending with $x_{50} = 355.33$. The following table gives the values of x_t and $x(t)$ for $t = 0, 5, 10, \ldots, 50$:

t	x_t	$x(t)$
0	50.00	50.00
5	60.83	61.07
10	74.01	74.59
15	90.05	91.11
20	109.56	111.28
25	133.29	135.91
30	162.17	166.01
35	197.30	202.76
40	240.05	247.65
45	292.06	302.48
50	355.33	369.45

Notice that the error in our approximation, that is, the difference between x_k and $x(k)$, increases as k increases. For example,

$$x(5) - x_5 = 0.24,$$

whereas

$$x(50) - x_{50} = 14.12.$$

This should be expected since the error of the approximation on each step is compounded by the errors made in each of the preceding steps. The only way we can control the amount of error in our approximations is to decrease the step size. For example, if we reduce the step size to $h = 0.1$, we obtain

$$x_1 = x_0 + h(0.04x_0) = 50 + (0.1)(0.04)(50) = 50 + 0.2 = 50.2000,$$
$$x_2 = x_1 + h(0.04x_1) = 50.2 + (0.1)(0.04)(50.2) = 50.2 + 0.2008 = 50.4008,$$
$$x_3 = x_2 + h(0.04x_2) = 50.4008 + (0.1)(0.04)(50.4008) = 50.4008 + 0.2016 = 50.6024,$$
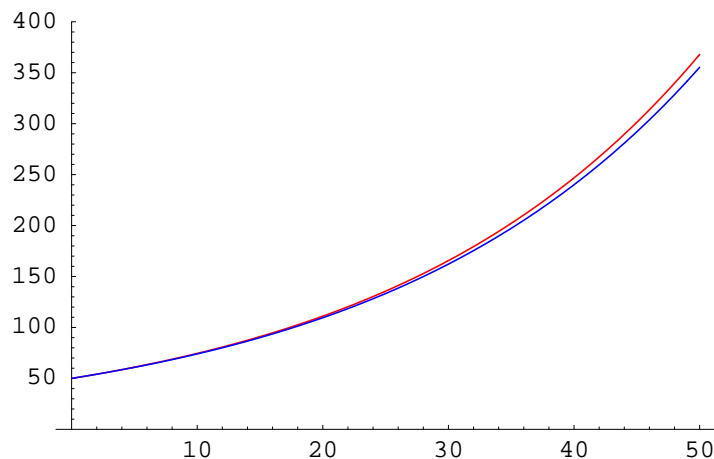
Figure 8.1.1 Approximate solutions of $\dot{x} = 0.04x$

and so on, ending with $x_{500} = 367.98$ as our approximation for $x(50)$. In general, with $h = 0.1$, x_k is an approximation for $x(0+kh) = x(0.1k)$. Equivalently, $x(t)$ is approximated by x_k where $k = 10t$, provided $10t$ is an integer. The following table gives the values of x_{10t} and $x(t)$ for $t = 0, 5, 10, \ldots, 50$:

t	x_{10t}	$x(t)$
0	50.00	50.00
5	61.05	61.07
10	74.53	74.59
15	91.00	91.11
20	111.10	111.28
25	135.64	135.91
30	165.61	166.01
35	202.19	202.76
40	246.86	247.65
45	301.40	302.48
50	367.98	369.45

As expected, reducing the step size from $h = 1$ to $h = 0.1$ greatly reduces the error in the approximations. Figure 8.1.1 shows the graphs of both our approximations to x (the graph of x is not shown since, on this scale, it is essentially the same as our second approximation).

Example   The differential equation in the previous example could be used to model a population which is growing at a rate of approximately 4% per unit of time, starting with an initial population of 50. As a modification of this model, consider a case where the rate of growth of the population is decreasing over time. For example, the rate of growth might start out at 4% but decrease over time toward 2%. If $x(t)$ is the population after t units of time, then the differential equation
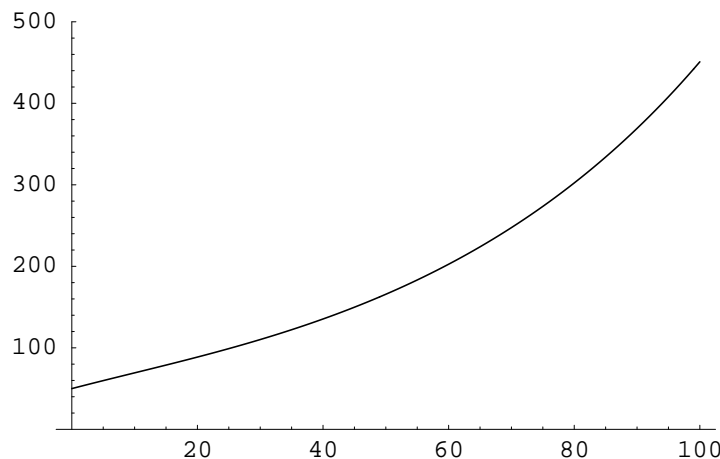
$$\dot{x} = 0.02(1 + e^{-\frac{t}{10}})x$$

Figure 8.1.2 An approximate solution of $\dot{x} = 0.02(1 + e^{-\frac{t}{10}})x$

would describe one such situation. To approximate the solution to this equation over the interval $[0, 100]$, again assuming an initial population of $x_0 = 50$, we will use Euler's method with $h = 0.05$ and compute $x_1, x_2, \ldots, x_{2000}$. Then, with

$$f(x, t) = 0.02(1 + e^{-\frac{t}{10}})x,$$

we have

$$x_1 = x_0 + hf(50, 0) = 50 + (0.05)(2) = 50 + 0.1 = 50.10000,$$
$$x_2 = x_1 + hf(50.1, 0.05) = 50.1 + (0.05)(1.99900) = 50.1 + 0.09995 = 50.19995,$$
$$x_3 = x_2 + hf(50.19995, 0.10) = 50.19995 + (0.05)(1.99801)$$
$$= 50.19995 + 0.09990 = 50.29985,$$

and so on, ending with $x_{2000} = 450.91$ as our approximation to $x(20)$, where we have rounded x_1, x_2, and x_3 to five decimal places and x_{2000} to two decimal places. In general, x_k is an approximation for $x(0 + 0.05k) = x(0.05k)$; that is, $x(t)$ is approximated by x_{20t}. The following table lists the values of x_{20t}, rounded to two decimal places, for $t = 0, 10, 20, \ldots, 100$:

t	x_{20t} $(\approx x(t))$
0	50.00
10	69.30
20	88.67
30	110.17
40	135.40
50	165.74
60	202.59
70	247.50
80	302.30
90	369.20
100	450.91

The graph of our approximate solution is shown in Figure 8.1.2.

As we have seen, the accuracy of Euler's method depends on the step size h. In theory, we can obtain any level of accuracy we desire by choosing h small enough; however, in practice there are limitations on how small we can make h. These limitations include the level of precision with which a given computer represents numbers, the time it takes to perform the necessary computations (smaller values of h require more iterations of (8.1.10)), and the accumulation of round-off error resulting from requiring large numbers of iterations. Fortunately, there are many ways to improve upon Euler's method, one of which we will consider now.

For the equation

$$\dot{x} = f(x,t)$$

with $x(t_0) = x_0$, Euler's method is based on using $hf(x_0, t_0)$ as an approximation to the difference $x(t_0 + h) - x_0$. The accuracy of this approximation will depend upon how much $\dot{x}(t)$ differs from $f(x_0, t_0)$ over the interval $[t_0, t_0 + h]$. In general, the accuracy of our approximation will improve if we use

$$\dot{x}\left(t_0 + \frac{h}{2}\right),$$

the slope of x at the midpoint of the interval $[t_0, t_0 + h]$, in place of $\dot{x}(t_0) = f(x_0, t_0)$, the slope of x at the left-hand endpoint of this interval. Now

$$\dot{x}\left(t_0 + \frac{h}{2}\right) = f\left(x\left(t_0 + \frac{h}{2}\right), t_0 + \frac{h}{2}\right). \tag{8.1.11}$$

However, we do not know $x\left(t_0 + \frac{h}{2}\right)$. To get around this difficulty, we will first use Euler's method to approximate $x\left(t_0 + \frac{h}{2}\right)$ by

$$x_0 + \frac{h}{2}f(x_0, t_0),$$

and then approximate $\dot{x}\left(t_0 + \frac{h}{2}\right)$ by

$$f\left(x_0 + \frac{h}{2}f(x_0, t_0), t_0 + \frac{h}{2}\right). \tag{8.1.12}$$

Replacing $f(x_0, t_0)$ by (8.1.12) in Euler's method, we have

$$x_1 = x_0 + hf\left(x_0 + \frac{h}{2}f(x_0, t_0), t_0 + \frac{h}{2}\right) \tag{8.1.13}$$

as our approximation for $x(t_0 + h)$. It can be shown that in this case

$$x(t_0 + h) - x_1 = O(h^3),$$

whereas we have seen that the error in one step of Euler's method is $O(h^2)$. This method of approximation is known as the *Runge-Kutta method of order 2*, after the mathematicians

Carl Runge (1856-1927) and M. W. Kutta (1867-1944). In general, an approximation method is said to be of order n if the error in one step is $O(h^{n+1})$. There are Runge-Kutta formulas for approximations of higher order, but we will consider only this order 2 formula.

Second order Runge-Kutta   To approximate the solution of the equation

$$\dot{x} = f(x, t)$$

with initial condition $x(t_0) = x_0$ on an interval $[t_0, t_1]$, choose a small value for $h > 0$ and an integer n such that $t_0 + nh \geq t_1$. Letting $s_k = t_0 + kh$, $k = 0, 1, 2, \ldots n$, compute

$$m = \frac{h}{2} f(x_k, s_k)$$

and

$$x_{k+1} = x_k + hf\left(x_k + m, s_k + \frac{h}{2}\right) \tag{8.1.14}$$

for $k = 0, 1, \ldots n - 1$. Then x_k is an approximation for $x(t_0 + kh)$.

Example   We will approximate the solution of $\dot{x} = 0.04x$ with $x(0) = 50$ on the interval $[0, 50]$ using the second order Runge-Kutta method with $h = 0.1$. Using $f(x, t) = 0.04x$ and rounding to four decimal places, we have

$$
\begin{aligned}
x_1 &= 50 + 0.1 f(50 + 0.05 f(50, 0), 0.05) \\
&= 50 + 0.1 f(50.1, 0.05) \\
&= 50 + 0.2004 \\
&= 50.2004, \\
x_2 &= 50.2004 + 0.1 f(50.2004 + 0.05 f(50.2004, 0.1), 0.15) \\
&= 50.2004 + 0.1 f(50.3008, 0.15) \\
&= 50.2004 + 0.2012 \\
&= 50.4016,
\end{aligned}
$$

and so on, up to $x_{500} = 369.4508$. Since $h = 0.1$, x_{10t} gives us an approximation for $x(t)$ whenever $10t$ is an integer. The following table gives the values of x_{10t} and $x(t)$ for $t = 0, 5, 10, \ldots, 50$:

t	x_{10t}	$x(t)$
0	50.0000	50.0000
5	61.0701	61.0701
10	74.5912	74.5912
15	91.1058	91.1059
20	111.2768	111.2770
25	135.9137	135.9141
30	166.0053	166.0058
35	202.7592	202.7600
40	247.6506	247.6516
45	302.4809	302.4824
50	369.4508	369.4528

Comparing these values with the values we obtained with Euler's method for the same equation, we see that we have decreased our error significantly without decreasing the step size. Hence we have gained accuracy without greatly increasing the number of computations required.

Example In Problem 7 in Section 4.8 (which is repeated in Problem 2 at the end of this section) we discussed the problem of a body in free fall near the surface of the earth, neglecting the effects of air resistance. In that case, if the body has mass m and F is the force acting on the body, then, since we are neglecting all forces except for that of gravity, $F = -mg$, where $g = 32$ feet/second^2. However, if we consider the effects of air resistance, then we have to modify F to account for this additional force acting in a direction opposite to that of gravity. One common assumption is that air resistance is proportional to velocity; in that case, we have

$$F = -32m - kv, \tag{8.1.15}$$

where $k > 0$ is a constant which depends on the particular body and v is the velocity of the body. Note that $kv < 0$ since $v < 0$, so the additional force $-kv$ is acting in the opposite direction of gravity. Now if a is the acceleration of the body, $F = ma$ implies

$$ma = -32m - kv, \tag{8.1.16}$$

and so

$$a = -32 - \frac{k}{m}v. \tag{8.1.17}$$

Since $a = \dot{v}$, (8.1.17) gives us the differential equation

$$\dot{v} = -32 - cv, \tag{8.1.18}$$

where

$$c = \frac{k}{m}$$

is a constant which depends both on the mass of the body and its air resistance. For example, suppose we have a situation where $c = 0.1$ and the body is released from rest high above the ground. Then we are interested in the solution of the equation

$$\dot{v} = -32 - 0.1v$$

with the initial condition $v(0) = 0$. The following table gives the results from applying the second order Runge-Kutta method with step size $h = 0.1$ over the interval $[0, 70]$:

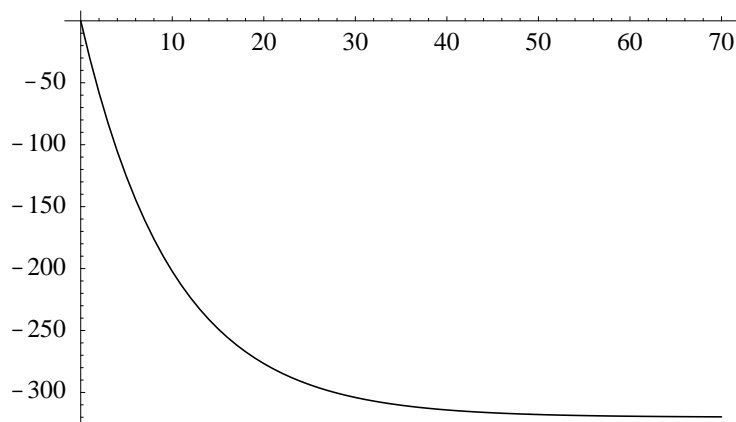| t	v_{10t} $(\approx v(t))$
0	0.0
5	-125.9
10	-202.3
15	-248.6
20	-276.7
25	-293.7
30	-304.1
35	-310.3
40	-314.1
45	-316.4
50	-317.8
55	-318.7
60	-319.2
65	-319.5
70	-319.7

Figure 8.1.3 Velocity of a body in free fall

From the table of values and from the graph of the approximate solution in Figure 8.1.3, it appears that the velocity of the body approaches a limiting value. We shall see in the next section that this is indeed the case. For this example, the velocity will approach -320 feet per second. We call this velocity the *terminal velocity* of the body.

Problems

1. Solve each of the following differential equations using the given initial condition.

   (a) $\dot{x} = t^2 - 2$, $x(0) = 3$ (b) $\dot{x} = -\sin(t)$, $x(0) = 2$

   (c) $\dot{y} = \sqrt{t}$, $y(1) = -3$ (d) $\dot{w} = te^{-t}$, $w(0) = 2$

2. Let $x(t)$, $v(t)$, and $a(t)$ be the height, velocity, and acceleration, respectively, of an object of mass m in free fall near the surface of the earth. Let x_0 and v_0 be the

height and velocity, respectively, of the object at time t_0. If we ignore the effects of air resistance, the force acting on the body is $-mg$, where g is a constant ($g = 9.8$ meters/second^2 or $g = 32$ feet/second^2). Thus by Newton's second law of motion,

$$-mg = ma(t),$$

and so

$$a(t) = -g.$$

Show that

$$x(t) = -\frac{1}{2}gt^2 + v_0t + x_0.$$

3. Suppose an object is projected vertically upward from a height of 100 feet with an initial velocity of 20 feet per second. Use Problem 2 to answer the following questions.

    (a) Find $x(t)$, the height of the object at time t.
    (b) At what time does the object reach its maximum height?
    (c) What is the maximum height reached by the object?
    (d) At what time will the object strike the ground?

4. For each of the following, use Euler's method to approximate the solution of the equation over the given interval I using the step size h specified. Plot your result.

    (a) $\dot{x} = e^{-2t^2}$, $x(0) = 1$, $I = [0, 10]$, $h = 0.1$
    (b) $\dot{x} = -0.96x$, $x(0) = 100$, $I = [0, 100]$, $h = 0.5$
    (c) $\dot{y} = 0.05y - 5$, $y(0) = 200$, $I = [0, 50]$, $h = 0.05$
    (d) $\dot{x} = 0.002x(100 - x)$, $x(0) = 10$, $I = [0, 200]$, $h = 0.1$
    (e) $\dot{y} = 0.02(1 + 0.5e^{-\frac{t}{15}})y$, $y(0) = 10$, $I = [0, 150]$, $h = 0.5$
    (f) $\dot{x} = \cos(x^2)$, $x(0) = 0$, $I = [0, 10]$, $h = 0.01$
    (g) $\dot{x} = 0.045x + 0.4t$, $x(0) = 30$, $I = [0, 40]$, $h = 0.1$
    (h) $\dot{x} = 0.1x + \cos(t)$, $x(0) = 0$, $I = [0, 10]$, $h = 0.05$

5. Use the second order Runge-Kutta method to approximate solutions to the equations in Problem 4.

6. In 1990 the population of India was 853.4 million. If $P(t)$ is the population of India t years after 1990, suppose P satisfies the differential equation

$$\dot{P} = k(t)P,$$

where $k(t)$ is the rate of growth of the population at time t. For example, at the start of 1990 the population of India was growing at a rate of 2.1% per year, so $k(0) = 0.021$.

    (a) Suppose the rate of growth remains constant; that is, suppose $k(t) = 0.021$ for all $t \geq 0$. Find $P(t)$. In what year will the population reach 1 billion? In what year will it reach 2 billion? In what year will it reach 3 billion?

(b) Now suppose $k(t)$ is decreasing toward 1% in such a way that

$$k(t) = 0.01(1 + 1.1e^{-\frac{t}{30}}).$$

Use the second order Runge-Kutta method to approximate P over the interval $[0, 100]$ using a step size of $h = 0.1$. In what year will the population reach 1 billion? In what year will it reach 2 billion? In what year will it reach 3 billion?

(c) Plot your results from (a) and (b) together.

7. In the final example in this section we considered the problem of an object in free fall when the air resistance is proportional to the velocity of the object. Now consider the case where the air resistance is proportional to the square root of the speed.

(a) If s is the speed of the object, in feet per second, t seconds after it is released, explain why s satisfies the differential equation

$$\dot{s} = 32 - c\sqrt{s},$$

$s(0) = 0$, for some constant $c > 0$.

(b) Using $c = 1$, use the second order Runge-Kutta method to solve the equation in (a) over the interval $[0, 500]$ using a step size of $h = 0.5$. Plot your results.

(c) Does the object appear to approach a limiting speed? If so, what is the terminal speed?

(d) Solve the equation

$$32 - c\sqrt{s} = 0$$

for s. Explain the connection between this answer and your answer in (c).

8. In the final example in this section we considered the problem of an object in free fall when the air resistance is proportional to the velocity of the object. Now consider the case where the air resistance is proportional to the square of the velocity.

(a) If v is the velocity of the object, in feet per second, t seconds after it is released, explain why v satisfies the differential equation

$$\dot{v} = -32 + cv^2,$$

$v(0) = 0$, for some constant $c > 0$.

(b) Using $c = 0.01$, use the second order Runge-Kutta method to approximate the solution to the equation in (a) over the interval $[0, 20]$ using a step size of $h = 0.02$. Plot your results.

(c) Does the object appear to approach a limiting velocity? If so, what is the terminal velocity?

(d) Solve the equation

$$-32 + cv^2 = 0$$

for v. Explain the connection between this answer and your answer in (c).

9. Suppose the population of a certain country was 56 million in 2000 and the natural rate of the growth of the population was 2% per year. Moreover, suppose $k(t)$ represents the net rate of growth of the population due to immigration and emigration t years after 2000.

(a) Let $P(t)$ be the population of the country t years after 2000. Explain why P should satisfy the differential equation

$$\dot{P} = 0.02P + k(t),$$

with $P(0) = 56$.

(b) If $k(t) = 0.04t$, use the second order Runge-Kutta method to approximate the solution to the equation in (a) over the interval $[0, 25]$ using a step size of $h = 0.05$. Plot your results.

(c) What does this model predict for the population of the country in the year 2010?

(d) When will the population of the country reach 100 million?