Difference Equations
to
Differential Equations

Section 7.4

The Two-Body Problem

In 1609 Johann Kepler (1571-1630) published the first two of his three laws of planetary motion. The first of these states that the orbit of a planet about the sun is an ellipse with the sun at one focus. He had reached this conclusion after painstaking analysis of the data Tycho Brahe (1546-1601) had collected from observing the motion of Mars over a period of more than 20 years. His work was a scientific triumph because it created a model for the solar system that was not only more accurate than the models of Copernicus and Ptolemy, but simpler as well. Yet, however brilliant, Kepler's result amounted to fitting a curve to a set of data without discovering any fundamental principles underlying the motion of planets that would cause their orbits to be as we observe them. In 1687 Newton provided the missing principles. In his great work, *Philosophiae naturalis principia mathematica*, Newton demonstrated that the elliptical orbit of a planet is a consequence of his three laws of motion and the inverse square law of gravitation. Hence the behavior of the planets could be explained by the same laws which govern the path of an apple as it falls from a tree to the ground; for the first time it became clear that the so-called heavenly bodies behaved no differently than the seemingly more substantial bodies of our everyday experience.

In this section we will see how the motion of the planets may be explained using only Newton's laws and tools from our study of calculus. The solution of this problem is one of the greatest triumphs of the human intellect in general and of calculus in particular.

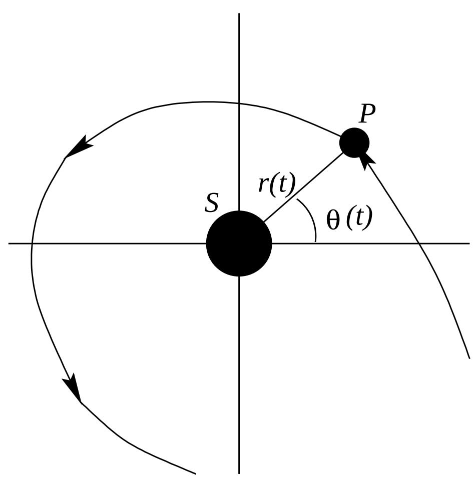

Figure 7.4.1 Possible orbit of a body P about a body S

To begin, suppose we have two bodies, one of mass m, which we denote by P, and the other of mass M, which we denote by S. We may think of S as representing the sun and

P as representing a planet. It is possible to show that Newton's laws of motion hold in a coordinate system with the origin located at the center of mass of the two bodies; for simplicity, we will assume that M is significantly larger than m (as it is if S is the sun and P is a planet, asteroid, or comet), allowing us to assume that the center of mass is located at S. Thus we choose a coordinate system for the complex plane so that S is at the origin and we let $z(t)$ represent the position of P with respect to S at time t. If we express $z(t)$ in polar coordinates, then

$$z(t) = r(t)e^{i\theta(t)}, \tag{7.4.1}$$

where r and θ are real-valued functions, as shown in Figure 7.4.1. For simplicity of notation, we will usually drop the explicit reference to t and simply write

$$z = re^{i\theta}. \tag{7.4.2}$$

By Newton's law of gravitation the magnitude of the gravitational force of attraction between the two bodies is

$$|F| = \frac{GMm}{r^2}, \tag{7.4.3}$$

where G is a constant, approximately

$$6.67 \times 10^{-11} \frac{\text{N m}^2}{\text{kg}^2}$$

if we measure force in Newtons, distance in meters, and mass in kilograms. Since gravity is an attractive force and we are assuming S to be at rest at the origin, F is directed from P toward the origin. Hence we have

$$F = -\frac{GMm}{r^2}e^{i\theta}. \tag{7.4.4}$$

Moreover, we assume that this is the only force acting on the two bodies. Now if $v(t)$ and $a(t)$ represent the velocity and acceleration, respectively, of P at time t, then, by Newton's second law of motion, $F = ma$, we must have

$$ma = -\frac{GMm}{r^2}e^{i\theta}. \tag{7.4.5}$$

Letting $k = GM$, this simplifies to

$$a = -\frac{k}{r^2}e^{i\theta}. \tag{7.4.6}$$

From our work in Section 7.3 we know that

$$v = \frac{dz}{dt} = \frac{d}{dt}re^{i\theta} = ire^{i\theta}\frac{d\theta}{dt} + e^{i\theta}\frac{dr}{dt} \tag{7.4.7}$$

and

$$\begin{aligned}
a &= \frac{dv}{dt} \\
&= \frac{d}{dt}\left(ire^{i\theta}\frac{d\theta}{dt} + e^{i\theta}\frac{dr}{dt}\right) \\
&= ire^{i\theta}\frac{d}{dt}\left(\frac{d\theta}{dt}\right) + \frac{d\theta}{dt}\frac{d}{dt}(ire^{i\theta}) + e^{i\theta}\frac{d}{dt}\left(\frac{dr}{dt}\right) + \frac{dr}{dt}\frac{d}{dt}(e^{i\theta}) \\
&= ire^{i\theta}\frac{d^2\theta}{dt^2} + \frac{d\theta}{dt}\left(i^2re^{i\theta}\frac{d\theta}{dt} + ie^{i\theta}\frac{dr}{dt}\right) + e^{i\theta}\frac{d^2r}{dt^2} + \frac{dr}{dt}ie^{i\theta}\frac{d\theta}{dt} \\
&= ire^{i\theta}\frac{d^2\theta}{dt^2} - re^{i\theta}\left(\frac{d\theta}{dt}\right)^2 + ie^{i\theta}\frac{d\theta}{dt}\frac{dr}{dt} + e^{i\theta}\frac{d^2r}{dt^2} + ie^{i\theta}\frac{d\theta}{dt}\frac{dr}{dt} \\
&= -re^{i\theta}\left(\frac{d\theta}{dt}\right)^2 + e^{i\theta}\frac{d^2r}{dt^2} + i\left(re^{i\theta}\frac{d^2\theta}{dt^2} + 2e^{i\theta}\frac{d\theta}{dt}\frac{dr}{dt}\right). \qquad (7.4.8)
\end{aligned}$$

Putting (7.4.6) and (7.4.8) together gives us

$$-\frac{k}{r^2}e^{i\theta} = -re^{i\theta}\left(\frac{d\theta}{dt}\right)^2 + e^{i\theta}\frac{d^2r}{dt^2} + i\left(re^{i\theta}\frac{d^2\theta}{dt^2} + 2e^{i\theta}\frac{d\theta}{dt}\frac{dr}{dt}\right).$$

After dividing through by $e^{i\theta}$ we have

$$-\frac{k}{r^2} = -r\left(\frac{d\theta}{dt}\right)^2 + \frac{d^2r}{dt^2} + i\left(r\frac{d^2\theta}{dt^2} + 2\frac{d\theta}{dt}\frac{dr}{dt}\right). \qquad (7.4.9)$$

The equality in (7.4.9) implies that the the real part of the left-hand side of the equation is equal to the real part of the right-hand side of the equation and the imaginary part of the left-hand side of the equation is equal to the imaginary part of the right-hand side of the equation. That is,

$$-\frac{k}{r^2} = -r\left(\frac{d\theta}{dt}\right)^2 + \frac{d^2r}{dt^2} \qquad (7.4.10)$$

and

$$0 = r\frac{d^2\theta}{dt^2} + 2\frac{d\theta}{dt}\frac{dr}{dt}. \qquad (7.4.11)$$

Multiplying both sides of (7.4.11) by r gives us

$$0 = r^2\frac{d^2\theta}{dt^2} + 2r\frac{d\theta}{dt}\frac{dr}{dt}. \qquad (7.4.12)$$

However,

$$\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = r^2\frac{d^2\theta}{dt^2} + 2r\frac{d\theta}{dt}\frac{dr}{dt},$$

so (7.4.12) implies that

$$\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0. \tag{7.4.13}$$

Since a function with 0 for its derivative must be a constant function, it follows that

$$r^2\frac{d\theta}{dt} = c \tag{7.4.14}$$

for some constant c. In any interval of time of interest, we will have $r > 0$, that is, S and P are not a the same point in space, and so $r^2 > 0$. It follows that if $c = 0$, then $\frac{d\theta}{dt} = 0$ for all t, corresponding to the relatively uninteresting case when θ is a constant and P moves along a straight line passing through S. The more interesting cases are when $c < 0$ or $c > 0$. Since the former case implies that $\frac{d\theta}{dt} < 0$ for all t and the latter implies $\frac{d\theta}{dt} > 0$ for all t, the choice of sign for c ultimately depends on our choice of orientation in our coordinate system, that is, the direction in which we measure positive angles. Hence, without loss of generality, we may assume $c > 0$ or, equivalently, $\frac{d\theta}{dt} > 0$.

We will now use the substitution $s = \frac{1}{r}$ to put (7.4.10) into a simpler form. With this substitution, $r = \frac{1}{s}$, so

$$\frac{dr}{dt} = \frac{d}{dt}\left(\frac{1}{s}\right) = -\frac{1}{s^2}\frac{ds}{dt} = -\frac{1}{s^2}\frac{ds}{d\theta}\frac{d\theta}{dt}. \tag{7.4.15}$$

Since, from (7.4.14),

$$\frac{d\theta}{dt} = \frac{c}{r^2} = cs^2, \tag{7.4.16}$$

we have

$$\frac{dr}{dt} = -c\frac{ds}{d\theta}. \tag{7.4.17}$$

Differentiating again,

$$\frac{d^2r}{dt^2} = \frac{d}{dt}\left(-c\frac{ds}{d\theta}\right) = -c\frac{d}{dt}\left(\frac{ds}{d\theta}\right) = -c\frac{d}{d\theta}\left(\frac{ds}{d\theta}\right)\frac{d\theta}{dt} = -c\frac{d\theta}{dt}\frac{d^2s}{d\theta^2}. \tag{7.4.18}$$

Hence, using (7.4.16),

$$\frac{d^2r}{dt^2} = -c^2s^2\frac{d^2s}{d\theta^2}. \tag{7.4.19}$$

Finally, substituting (7.4.16), (7.4.19), and $s = \frac{1}{r}$ into (7.4.10) gives us

$$-ks^2 = -\frac{1}{s}(cs^2)^2 - c^2s^2\frac{d^2s}{d\theta^2} = -c^2s^3 - c^2s^2\frac{d^2s}{d\theta^2}. \tag{7.4.20}$$

Dividing both sides of this equation by $-c^2 s^2$, we have

$$\frac{d^2 s}{d\theta^2} + s = \frac{k}{c^2}. \tag{7.4.21}$$

This is the differential equation to which all our work has been leading. The solution of this equation will be an expression for s as a function of θ; since r is in turn a function of s, namely, $r = \frac{1}{s}$, this will give us r as a function of θ and allow us to determine the path of motion of P. Note, however, that we will not have found r as a function of t. In other words, we will be able to determine the path of motion of P, but we will not be able to determine where along that path P is at any specific time t.

To solve (7.4.21), we first note that if $y(\theta)$ is a solution of the equation

$$\frac{d^2 y}{d\theta^2} + y = 0,$$

then the function

$$x(\theta) = y(\theta) + \frac{k}{c^2}$$

satisfies the equation

$$\frac{d^2 x}{d\theta^2} + x = \frac{k}{c^2}$$

since

$$\frac{d^2}{d\theta^2}\left(y + \frac{k}{c^2}\right) + \left(y + \frac{k}{c^2}\right) = \frac{d^2 y}{d\theta^2} + y + \frac{k}{c^2} = 0 + \frac{k}{c^2} = \frac{k}{c^2}.$$

Hence to solve (7.4.21), we need only solve the equation

$$\frac{d^2 s}{d\theta^2} + s = 0. \tag{7.4.22}$$

That is, we need only find a function s of θ such that

$$\frac{d^2 s}{d\theta^2} = -s. \tag{7.4.23}$$

Now (7.4.22) simply says that s is a function with the property that its second derivative is the negative of itself. But we already know two such functions, namely, $\sin(\theta)$ and $\cos(\theta)$; moreover, for any constants A and B, the function $A\sin(\theta) + B\cos(\theta)$ also has this property. Although the justification is beyond our resources at this point, it is in fact true that any solution of (7.4.23) must be of the form

$$A\sin(\theta) + B\cos(\theta) \tag{7.4.24}$$

for some constants A and B. From this it now follows that our sought after solution to (7.4.21) must have the form

$$s = A\sin(\theta) + B\cos(\theta) + \frac{k}{c^2} \tag{7.4.25}$$

for some constants A and B.

We will now find values for the constants A and B so that

$$\left.\frac{ds}{d\theta}\right|_{\theta=0} = 0 \tag{7.4.26}$$

and

$$\left.\frac{d^2 s}{d\theta^2}\right|_{\theta=0} \leq 0. \tag{7.4.27}$$

Intuitively, this means we are looking for values which satisfy conditions for s to have a local maximum at $\theta = 0$. Equivalently, these conditions will hold if r has a local minimum at $\theta = 0$. We make think of this as choosing the constants A and B in such a way that P is closest to S when the path of P crosses the positive real axis. Now

$$\frac{ds}{d\theta} = A\cos(\theta) - B\sin(\theta),$$

so

$$\left.\frac{ds}{d\theta}\right|_{\theta=0} = A, \tag{7.4.28}$$

and

$$\frac{d^2 s}{d\theta^2} = -A\sin(\theta) - B\cos(\theta),$$

so

$$\left.\frac{d^2 s}{d\theta^2}\right|_{\theta=0} = -B. \qquad (7.4.29).$$

Hence the conditions (7.4.26) and (7.4.27) are satisfied if we set $A = 0$ and we require $B \geq 0$. In other words, the conditions (7.4.26) and (7.4.27) are satisfied by

$$s = B\cos(\theta) + \frac{k}{c^2}, \tag{7.4.30}$$

where $B \geq 0$.

In terms of r, (7.4.30) gives us

$$\frac{1}{r} = B\cos(\theta) + \frac{k}{c^2} = \frac{c^2 B\cos(\theta) + k}{c^2},$$

so

$$r = \frac{c^2}{c^2 B\cos(\theta) + k} = \frac{\dfrac{c^2}{k}}{1 + \dfrac{c^2 B}{k}\cos(\theta)}. \tag{7.4.31}$$

If we let $\alpha = \frac{c^2}{k}$ and $\epsilon = \alpha$B, then our expression for r as a function of θ reduces to

$$r = \frac{\alpha}{1 + \epsilon\cos(\theta)}, \tag{7.4.32}$$

where $\epsilon \geq 0$ and $\alpha > 0$ are constants.

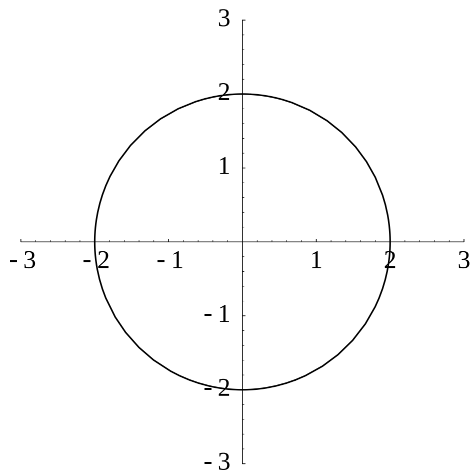

Figure 7.4.2 Circular orbit for P when $\epsilon = 0$ and $\alpha = 2$

Note that, as indicated above, our solution does not give us values of r and θ for specified values of t, but rather (7.4.32) gives us a value of r for any specified value of θ. In other words, our solution does not give us the position of P for a given time t, but it does tell us the location of P as a function of θ. Indeed, if we plot the points $z = re^{i\theta}$ for all values of θ in the interval $[-\pi, \pi]$, with r given by (7.4.32), then the resulting curve will be the path of the orbit of P about S. For example, if $\epsilon = 0$, then $r = \alpha$ for all t and the orbit of P is a circle of radius α with center at S, as shown in Figure 7.4.2 for $\alpha = 2$. Note that because of our assumption that $\frac{d\theta}{dt} > 0$, the motion along this curve, and all subsequent curves, will be in the counter-clockwise direction.

If $0 < \epsilon < 1$, then $\epsilon\cos(\theta)$ has a maximum value of ϵ when $\theta = 0$ and a minimum value of $-\epsilon$ when $\theta = -\pi$ or $\theta = \pi$. Thus the minimum value of r is

$$r(0) = \frac{\alpha}{1+\epsilon}$$

and the maximum value of r is

$$r(-\pi) = r(\pi) = \frac{\alpha}{1-\epsilon}.$$

Hence the orbit of P about S is a closed curve with

$$\frac{\alpha}{1+\epsilon} \leq r \leq \frac{\alpha}{1-\epsilon}$$

for all θ. An example for $\alpha = 2$ and $\epsilon = 0.5$, in which case $\frac{4}{3} \leq r \leq 4$ for all θ, is shown in Figure 7.4.3.

Note that

$$\lim_{\epsilon \to 1^-} r(0) = \lim_{\epsilon \to 1^-} \frac{\alpha}{1+\epsilon} = \frac{\alpha}{2},$$

whereas

$$\lim_{\epsilon \to 1^-} r(\pi) = \lim_{\epsilon \to 1^-} \frac{\alpha}{1-\epsilon} = \infty.$$

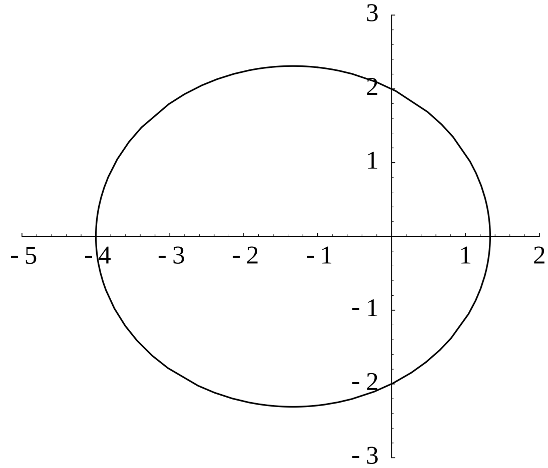

Figure 7.4.3 Orbit of P for $\epsilon = 0.5$ and $\alpha = 2$

Hence as ϵ approaches 1 from the left, the point of closest approach of P to S shrinks toward $\frac{\alpha}{2}$, but the point at which P is farthest from S increases without bound. Thus, as ϵ varies from 0 to 1, the orbit of P flattens out, changing from a circle to a long oblong shape. Figure 7.4.4 shows the orbit of P for α =2 and $\epsilon = 0.95$, in which case $1.026 \leq r \leq 40$ for all θ . Because of this behavior, ϵ is called the *eccentricity* of the orbit of P.

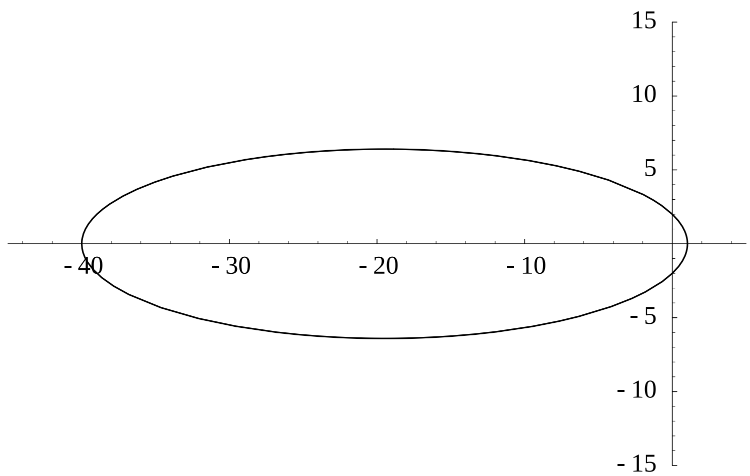

Figure 7.4.4 Orbit of P for $\epsilon = 0.95$ and $\alpha = 2$

When $\epsilon = 1$, r is not defined for $\theta = -\pi$ and $\theta = \pi$. In fact, in this case

$$\lim_{\theta \to \pi^-} r(\theta) = \lim_{\theta \to \pi^-} \frac{\alpha}{1 + \cos(\theta)} = \infty$$

and

$$\lim_{\theta \to -\pi^+} r(\theta) = \lim_{\theta \to -\pi^+} \frac{\alpha}{1 + \cos(\theta)} = \infty.$$

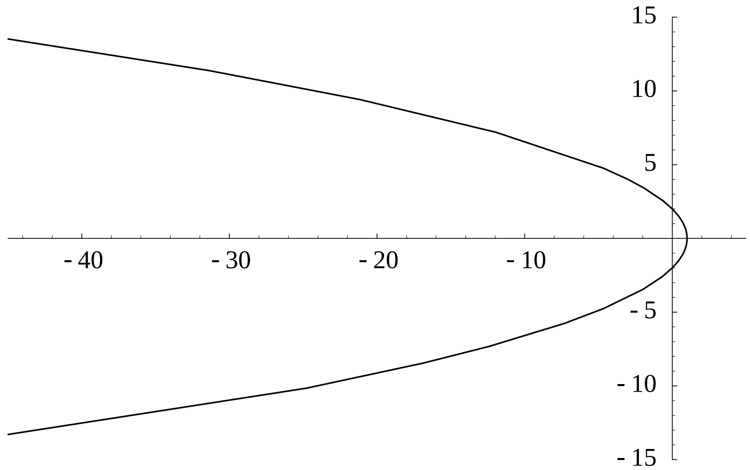

Figure 7.4.5 Orbit of P for $\epsilon = 1$ and $\alpha = 2$

Hence the orbit of P is not closed; P makes its closest approach to S when $\theta = 0$, at which point the distance from P to S is $\frac{\alpha}{2}$, and then follows a path which takes it ever farther away from S. The situation for $\alpha = 2$ and $\epsilon = 1$ is shown in Figure 7.4.5.

For $\epsilon > 1$, there are angles θ_1 and θ_2, with

$$-\pi < \theta_1 < -\frac{\pi}{2}$$

and

$$\frac{\pi}{2} < \theta_2 < \pi,$$

such that

$$\cos(\theta_1) = \cos(\theta_2) = -\frac{1}{\epsilon}. \tag{7.4.33}$$

Whenever $-\pi \leq \theta \leq \theta_1$ or $\theta_2 \leq \theta \leq \pi$ we have $1 + \epsilon\cos(\theta) \leq 0$. Since $\alpha > 0$ and $r \geq 0$ for all θ, the orbit of P in this case is defined by (7.4.32) only when $\theta_1 < \theta < \theta_2$. Moreover,

$$\lim_{\theta \to \theta_1^+} r(\theta) = \lim_{\theta \to \theta_1^+} \frac{\alpha}{1 + \epsilon\cos(\theta)} = \infty$$

and

$$\lim_{\theta \to \theta_2^-} r(\theta) = \lim_{\theta \to \theta_2^-} \frac{\alpha}{1 + \epsilon\cos(\theta)} = \infty$$

Thus again the orbit of P is not closed; P approaches S to within a distance of $\frac{\alpha}{1+\epsilon}$ at $\theta = 0$ and then follows a path away from S. See Figure 7.4.6 for the case $\alpha = 2$ and $\epsilon = 2$.

The curves in Figures 7.4.3 through 7.4.6 should look familiar. Indeed, the curves in Figures 7.4.3 and 7.4.4 are both ellipses, the curve in Figure 7.4.5 is a parabola, and the curve in Figure 7.4.6 is a hyperbola. This is not hard to see if we rewrite the equation

$$r = \frac{\alpha}{1 + \epsilon\cos(\theta)} \tag{7.4.34}$$

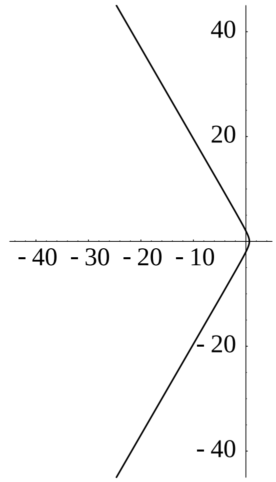

Figure 7.4.6 Orbit of P for $\epsilon = 2$ and $\alpha = 2$

in rectangular coordinates. Recall that if x and y are, respectively, the real and imaginary parts of $z = re^{i\theta}$, then

$$r = \sqrt{x^2 + y^2}$$

and

$$\cos(\theta) = \frac{x}{\sqrt{x^2 + y^2}}.$$

Hence if z is a point on the curve with equation (7.4.34), we have

$$\sqrt{x^2 + y^2} = \frac{\alpha}{1 + \dfrac{\epsilon x}{\sqrt{x^2 + y^2}}} = \frac{\alpha\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2} + \epsilon x}.$$

Dividing both sides by $\sqrt{x^2 + y^2}$ gives us

$$1 = \frac{\alpha}{\sqrt{x^2 + y^2} + \epsilon x},$$

and so

$$\sqrt{x^2 + y^2} = \alpha - \epsilon x.$$

Squaring, we have

$$x^2 + y^2 = \alpha^2 - 2\alpha\epsilon x + \epsilon^2 x^2,$$

from which we obtain

$$(1 - \epsilon^2)x^2 + y^2 + 2\alpha\epsilon x - \alpha^2 = 0. \tag{7.4.35}$$

Thus if the polar coordinates of z satisfy (7.4.34), then the rectangular coordinates of z must satisfy (7.4.35). Moreover, we know from analytic geometry that a curve in the plane with equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0, \tag{7.4.36}$$

where a, b, c, d, e, and f are all constants, is an ellipse if $b^2 - 4ac < 0$, a parabola if $b^2 - 4ac = 0$, and a hyperbola if $b^2 - 4ac > 0$. Because of this result, we call the number

$$D = b^2 - 4ac \tag{7.4.37}$$

the *discriminant* of (7.4.36). In the case of (7.4.35), we have

$$D = 0 - 4(1 - \epsilon^2) = -4(1 - \epsilon^2). \tag{7.4.38}$$

Thus $D < 0$ when $0 \leq \epsilon < 1$, $D = 0$ when $\epsilon = 1$, and $D > 0$ when $\epsilon > 1$. Since we have already seen that the orbit of P is a circle when $\epsilon = 0$ (a circle being a particular case of an ellipse), we now have the following classification of the orbit of P about S in terms of the eccentricity ϵ:

Eccentricity	**Orbit of P**
$\epsilon = 0$	Circle
$0 < \epsilon < 1$	Ellipse
$\epsilon = 1$	Parabola
$\epsilon > 1$	Hyperbola

Recall that, collectively, these curves are known as the conic sections.

We have seen that starting with the assumptions of Newton's law of gravitation and his second law of motion, we may conclude that the orbit of a body P about another body S must be a conic section. As great as Newton's accomplishment was, scientifically, mathematically, and philosophically, it is not the end of the story. The work we have done only accounts for the interaction of two bodies, isolated without any forces acting on them other than their mutual gravitational attraction. In reality, to model our entire solar system we would have to consider, at the minimum, the effects of the gravitational fields of the sun plus at least nine planets, as well as numerous moons, asteroids, and comets. Because of these other considerations, the orbits of the planets are not true ellipses, although, since by far the most dominant force acting on any one planet is the gravitational attraction between it and the sun, the deviation from elliptical paths is small. The problem of the motion of three or more bodies interacting under the influence of their mutual gravitational attraction has challenged mathematicians since the time of Newton. However, we now know that this problem, known as the *n-body problem*, cannot, in general, be solved exactly. Since the work of Henri Poincaré (1854-1912), advances on this problem have been directed toward qualitative and numerical descriptions of the orbits, not toward exact analytic solutions. In fact it was Poincaré who first showed that even in the case of only three bodies, the orbits can be highly complex, revealing a sensitivity to initial conditions that would make predictions about the future path of a given body effectively impossible. The work on this problem continues to the present.

Problems

1. The *perihelion* of the orbit of a planet is the point of the orbit which is closest to the sun. The following table gives the eccentricity and the distance from the sun at perihelion for each of the known planets in our solar system. Note the distances are given in astronomical units, where one astronomical unit is approximately 92.9 million miles, the mean distance from the earth to the sun.

Planet	**Eccentricity**	**Distance at Perihelion**
Mercury	0.21	0.31
Venus	0.01	0.72
Earth	0.02	0.98
Mars	0.09	1.38
Jupiter	0.05	4.95
Saturn	0.06	9.02
Uranus	0.05	18.3
Neptune	0.01	29.8
Pluto	0.25	29.8

(a) Plot the orbits of each of the planets.

(b) The aphelion of the orbit of a planet is the point of the orbit which is farthest from the sun. Find the distance of each planet from the sun at aphelion.

(c) Which orbits are closest to being circular? Which ones deviate the most from being circular?

(d) Plot the orbits of Neptune and Pluto together. How do they differ?

2. The orbit of the Comet Kohoutek has an eccentricity of 0.9999 and its distance from the sun at perihelion is 0.14 astronomical units. Plot the orbit of Comet Kohoutek and compare it with the orbit of Pluto from Problem 1. How far away from the sun is Comet Kohoutek at aphelion?

3. The orbit of Halley's comet has an eccentricity of 0.967 and its distance from the sun at perihelion is 0.59 astronomical units. Plot the orbit of Halley's comet and compare it with the orbits of Pluto and Comet Kohoutek as found in Problems 1 and 2. How far away from the sun is Halley's comet at aphelion?

4. The orbit of Encke's comet has an eccentricity of 0.847 and its distance from the sun at perihelion is 0.34 astronomical units. Plot the orbit of Encke's comet and compare it with the orbits of Pluto, Comet Kohoutek, and Halley's comet as found in Problems 1, 2, and 3. How far away from the sun is Encke's comet at aphelion?

5. (a) Use the information in Problem 1 to find the equation for the orbit of the earth in rectangular coordinates (that is, an equation of the form (7.4.35)).

   (b) Use your result from (a) and the techniques of Section 4.8 to find the length of the earth's orbit. Convert your answer into miles.

   (c) What is the average speed of the earth in miles per hour?

6. (a) Use the information in Problem 1 to find the equation for the orbit of Pluto in rectangular coordinates (that is, an equation of the form (7.4.35)).

   (b) Use your result from (a) and the techniques of Section 4.8 to find the length of Pluto's orbit. Convert your answer into miles.

   (c) What is the average speed of Pluto in miles per hour? You will need to know that it takes Pluto 248 years to complete one orbit about the sun.

7. To solve the two-body problem we had to solve a differential equation of the form

$$\frac{d^2y}{dt^2} = -y.$$

In this problem we consider the equation

$$\frac{d^2y}{dt^2} = y. \tag{7.4.39}$$

(a) Find two functions, $y_1(t)$ and $y_2(t)$, which satisfy (7.4.39) and are such that $y_2(t)$ is not a constant multiple of $y_1(t)$.

(b) Show that

$$y(t) = Ay_1(t) + By_2(t)$$

satisfies (7.4.39) for any constants A and B.

(c) Find a solution $y(t)$ of (7.4.39) such that $y(0) = 2$ and

$$\left.\frac{dy}{dt}\right|_{t=0} = 4.$$