

***Difference Equations
to
Differential Equations***

**Section 6.5
Inverse Trigonometric Functions**

In this section we will introduce the inverse trigonometric functions. We will begin with the inverse tangent function since, as indicated in Section 6.4, we need it to complete the story of the integration of rational functions.

Strictly speaking, the tangent function does not have an inverse. Recall that in order for a function f to have an inverse function, for every y in the range of f there must be exactly one x in the domain of f such that $f(x) = y$. This is false for the tangent function since, for example, both $\tan(0) = 0$ and $\tan(\pi) = 0$. In fact, since the tangent function is periodic with period π , if $\tan(x) = y$, then $\tan(x + n\pi) = y$ for any integer n . However, the tangent function is increasing on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, taking on every value in its range $(-\infty, \infty)$ exactly once. Hence we may define an inverse for the tangent function if we consider it with the restricted domain $(-\frac{\pi}{2}, \frac{\pi}{2})$. That is, we will define an inverse tangent function so that it takes on only values in $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Definition The *arc tangent function*, with value at x denoted by either $\arctan(x)$ or $\tan^{-1}(x)$, is the inverse of the tangent function with restricted domain $(-\frac{\pi}{2}, \frac{\pi}{2})$.

In other words, for $-\frac{\pi}{2} < y < \frac{\pi}{2}$,

$$y = \tan^{-1}(x) \text{ if and only if } \tan(y) = x. \quad (6.5.1)$$

For example, $\tan^{-1}(0) = 0$, $\tan^{-1}(1) = \frac{\pi}{4}$, and $\tan^{-1}(-1) = -\frac{\pi}{4}$. In particular, note that even though $\tan(\pi) = 0$, $\tan^{-1}(0) = 0$ since 0 is between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, but π is not between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

The domain of the arc tangent function is $(-\infty, \infty)$, the range of the tangent function, and the range of the arc tangent function is $(-\frac{\pi}{2}, \frac{\pi}{2})$, the domain of the restricted tangent function. Moreover, since

$$\lim_{x \rightarrow \frac{\pi}{2}^-} \tan(x) = \infty$$

and

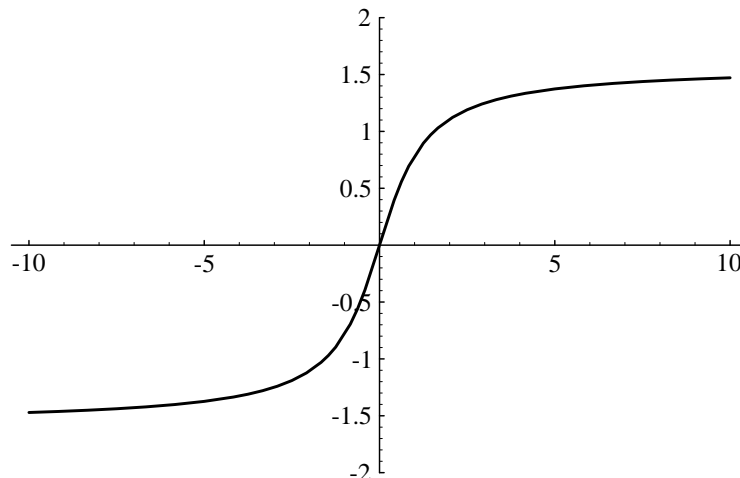
$$\lim_{x \rightarrow -\frac{\pi}{2}^+} \tan(x) = -\infty,$$

we have

$$\lim_{x \rightarrow \infty} \tan^{-1}(x) = \frac{\pi}{2} \quad (6.5.2)$$

and

$$\lim_{x \rightarrow -\infty} \tan^{-1}(x) = -\frac{\pi}{2}. \quad (6.5.3)$$

Figure 6.5.1 Graph of $y = \tan^{-1}(x)$

Hence $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$ are horizontal asymptotes for the graph of $y = \tan^{-1}(x)$, as shown in Figure 6.5.1.

To differentiate the arc tangent function we imitate the method we used to differentiate the logarithm function. Namely, if $y = \tan^{-1}(x)$, then $\tan(y) = x$, so

$$\frac{d}{dx} \tan(y) = \frac{d}{dx} x.$$

Hence

$$\sec^2(y) \frac{dy}{dx} = 1,$$

from which it follows that

$$\frac{dy}{dx} = \frac{1}{\sec^2(y)}.$$

Now

$$\sec^2(y) = 1 + \tan^2(y) = 1 + x^2,$$

so we have

$$\frac{dy}{dx} = \frac{1}{1 + x^2}.$$

Hence we have demonstrated the following proposition.

Proposition

$$\frac{d}{dx} \tan^{-1}(x) = \frac{1}{1 + x^2}. \quad (6.5.4)$$

As a consequence of the proposition, we also have

$$\int \frac{1}{1 + x^2} dx = \tan^{-1}(x) + c. \quad (6.5.5)$$

Note that $1 + x^2$ is an irreducible quadratic polynomial. We will see more examples of this type in the following examples.

Example Using the chain rule, we have

$$\frac{d}{dx} \tan^{-1}(4x^2) = \frac{8x}{1 + 16x^4}.$$

Example Evaluating $\int \tan^{-1}(x)dx$ is similar to evaluating $\int \log(x)dx$. That is, we will use integration by parts with

$$\begin{aligned} u &= \tan^{-1}(x) & dv &= dx \\ du &= \frac{1}{1+x^2} dx & v &= x. \end{aligned}$$

Then

$$\int \tan^{-1}(x)dx = x \tan^{-1}(x) - \int \frac{x}{1+x^2} dx.$$

Using the substitution

$$\begin{aligned} u &= 1 + x^2 \\ du &= 2x dx, \end{aligned}$$

we have $\frac{1}{2}du = x dx$, from which it follows that

$$\int \frac{x}{1+x^2} dx = \frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \log|u| + c = \frac{1}{2} \log(1+x^2) + c.$$

Thus

$$\int \tan^{-1}(x)dx = x \tan^{-1}(x) - \frac{1}{2} \log(1+x^2) + c.$$

Example To evaluate $\int \frac{1}{1+4x^2} dx$, we make the substitution

$$\begin{aligned} u &= 2x \\ du &= 2dx. \end{aligned}$$

Then $\frac{1}{2}du = dx$, so

$$\int \frac{1}{1+4x^2} dx = \frac{1}{2} \int \frac{1}{1+u^2} du = \frac{1}{2} \tan^{-1}(u) + c = \frac{1}{2} \tan^{-1}(2x) + c.$$

Example To evaluate $\int \frac{1}{x^2+x+1} dx$, we first note that x^2+x+1 does not factor, that is, is irreducible, and so we cannot use a partial fraction decomposition. In general,

a quadratic polynomial $ax^2 + bx + c$ is irreducible if $b^2 - 4ac < 0$ since, in that case, the quadratic formula yields complex solutions for the equation $ax^2 + bx + c = 0$. For $x^2 + x + 1$ we have $b^2 - 4ac = -3$. In this case it is helpful to simplify the function algebraically by completing the square of the denominator, thus making the problem similar to the previous example. That is, since

$$x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4},$$

we have

$$\int \frac{1}{x^2 + x + 1} dx = \int \frac{1}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} dx = \frac{4}{3} \int \frac{1}{\frac{4}{3}\left(x + \frac{1}{2}\right)^2 + 1} dx.$$

Now we can make the substitution

$$u = \sqrt{\frac{4}{3}} \left(x + \frac{1}{2}\right)$$

$$du = \sqrt{\frac{4}{3}} dx.$$

Then $\sqrt{\frac{3}{4}} du = dx$, so

$$\begin{aligned} \int \frac{1}{x^2 + x + 1} dx &= \frac{4}{3} \sqrt{\frac{3}{4}} \int \frac{1}{u^2 + 1} du \\ &= \sqrt{\frac{4}{3}} \tan^{-1}(u) + c \\ &= \sqrt{\frac{4}{3}} \tan^{-1} \left(\sqrt{\frac{4}{3}} \left(x + \frac{1}{2}\right) \right) + c. \end{aligned}$$

Partial fraction decomposition: Irreducible quadratic factors

The last two examples illustrate techniques that we may use to evaluate the integral of a rational function with an irreducible quadratic polynomial in the denominator. With this we are now in a position to consider the final case of partial fraction decomposition. Specifically, suppose we want to evaluate

$$\int \frac{f(x)}{g(x)} dx,$$

where f and g are both polynomials and the degree of f is less than the degree of g . Moreover, suppose that $(ax^2 + bx + c)^n$ is a factor of g , where n is a positive integer and

$ax^2 + bx + c$ is irreducible. Then the partial fraction decomposition of $\frac{f(x)}{g(x)}$ must contain a sum of terms of the form

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_nx + B_n}{(ax^2 + bx + c)^n}, \quad (6.5.6)$$

where A_1, A_2, \dots, A_n and B_1, B_2, \dots, B_n are constants. Note that the terms in the partial fraction decomposition corresponding to an irreducible quadratic factor differ from the terms for a linear factor in that the numerators of the terms in (6.5.6) need not be constants, but may be first degree polynomials themselves. As before, this is best illustrated with an example.

Example To evaluate $\int \frac{1+x}{x(1+x^2)} dx$ we need to find constants A , B , and C such that

$$\frac{1+x}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{1+x^2}.$$

Combining the terms on the right, we have

$$\frac{1+x}{x(1+x^2)} = \frac{A(1+x^2) + (Bx+C)x}{x(1+x^2)}.$$

Hence

$$1+x = A(1+x^2) + (Bx+C)x = (A+B)x^2 + Cx + A.$$

Equating the coefficients of the polynomials on the left and right gives us the system of equations

$$\begin{aligned} A+B &= 0, \\ C &= 1, \\ A &= 1. \end{aligned}$$

Thus $B = -1$ and

$$\frac{1+x}{x(1+x^2)} = \frac{1}{x} + \frac{1-x}{1+x^2} = \frac{1}{x} + \frac{1}{1+x^2} - \frac{x}{1+x^2}.$$

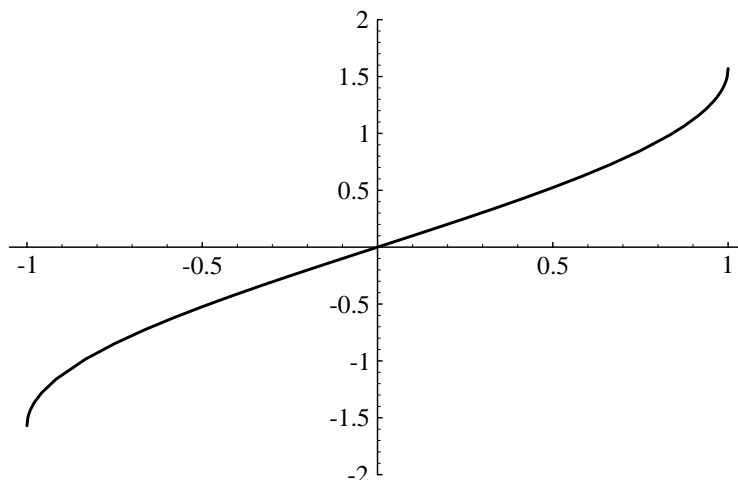
Hence

$$\begin{aligned} \int \frac{1+x}{x(1+x^2)} dx &= \int \frac{1}{x} dx + \int \frac{1}{1+x^2} dx - \int \frac{x}{1+x^2} dx \\ &= \log|x| + \tan^{-1}(x) - \frac{1}{2} \log(1+x^2) + c. \end{aligned}$$

where the final integral follows from the substitution $u = 1+x^2$ as in an earlier example.

If, unlike this example, the partial fraction decomposition of $\frac{f(x)}{g(x)}$ results in a term of the form

$$\frac{Ax+B}{(ax^2+bx+c)^n},$$

Figure 6.5.2 Graph of $y = \sin^{-1}(x)$

where $n > 1$ and $ax^2 + bx + c$ is irreducible, then the integration may still be difficult to carry out, perhaps even requiring some of the ideas of trigonometric substitutions that we will discuss in the next section. However, there is a limit to what should be done without the aid of a computer, or at least a table of integrals. There is a point after which some integrations become so complicated and time-consuming that in practice they should be given to a computer algebra system.

The inverse sine function

The remaining trigonometric functions all have inverses when their domains are restricted to appropriate intervals. Since the sine function is increasing on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, taking on every value in its range $[-1, 1]$ exactly once, we obtain an inverse for the sine function by restricting its domain to $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Definition The *arc sine function*, with value at x denoted by either $\arcsin(x)$ or $\sin^{-1}(x)$, is the inverse of the sine function with restricted domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

In other words, for $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$,

$$y = \sin^{-1}(x) \text{ if and only if } \sin(y) = x. \quad (6.5.7)$$

For example, $\sin^{-1}(0) = 0$, $\sin^{-1}(\frac{1}{2}) = \frac{\pi}{6}$, $\sin^{-1}(1) = \frac{\pi}{2}$, and $\sin^{-1}(-1) = -\frac{\pi}{2}$. Note that the domain of the arc sine function is $[-1, 1]$, the range of the sine function, and the range of the arc sine function is $[-\frac{\pi}{2}, \frac{\pi}{2}]$, the domain of the restricted sine function. The graph of $y = \sin^{-1}(x)$ is shown in Figure 6.5.2.

To find the derivative of the arc sine function, let $y = \sin^{-1}(x)$. Then $\sin(y) = x$, so

$$\frac{d}{dx} \sin(y) = \frac{d}{dx} x.$$

Hence

$$\cos(y) \frac{dy}{dx} = 1,$$

and so

$$\frac{dy}{dx} = \frac{1}{\cos(y)}.$$

Now

$$\cos^2(y) = 1 - \sin^2(y) = 1 + x^2,$$

so $\cos(y) = \pm\sqrt{1-x^2}$. Since $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, $\cos(y) \geq 0$. Thus $\cos(y) = \sqrt{1-x^2}$, and

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}.$$

Proposition

$$\frac{d}{dx} \sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}. \quad (6.5.8)$$

As a consequence of this proposition, we also have

$$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1}(x) + c. \quad (6.5.9)$$

Example Using the product and chain rules,

$$\frac{d}{dx} (x \sin^{-1}(2x)) = \frac{2x}{\sqrt{1-4x^2}} + \sin^{-1}(2x).$$

Example To evaluate $\int \frac{1}{\sqrt{4-x^2}} dx$, we first note that

$$\frac{1}{4-x^2} = \frac{1}{\sqrt{4(1-\frac{x^2}{4})}} = \frac{1}{2} \frac{1}{\sqrt{1-\frac{x^2}{4}}}.$$

Then the substitution

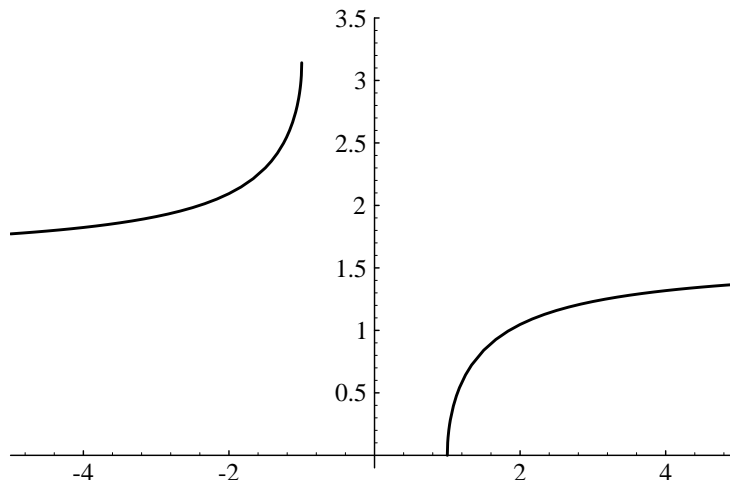
$$\begin{aligned} u &= \frac{x}{2} \\ du &= \frac{1}{2} dx \end{aligned}$$

gives us

$$\int \frac{1}{\sqrt{4-x^2}} dx = \int \frac{1}{\sqrt{1-u^2}} du = \sin^{-1}(u) + c = \sin^{-1}\left(\frac{x}{2}\right) + c.$$

The inverse secant function

Defining an inverse for the secant function is slightly more complicated than defining the arc tangent or arc sine functions. On the interval $[0, \frac{\pi}{2})$, the secant function is increasing

Figure 6.5.3 Graph of $y = \sec^{-1}(x)$

and takes on all values in the interval $[1, \infty)$; on the interval $(\frac{\pi}{2}, \pi]$, the secant function is also increasing, taking on all values in the interval $(-\infty, 1]$. Hence between these two intervals the secant function takes on every value in its range exactly once. From these considerations we obtain the following definition.

Definition The *arc secant function*, with value at x denoted by either $\operatorname{arcsec}(x)$ or $\sec^{-1}(x)$, is the inverse of the secant function with domain restricted to the intervals $[0, \frac{\pi}{2})$ and $(\frac{\pi}{2}, \pi]$.

Thus for $0 \leq y < \frac{\pi}{2}$ or $\frac{\pi}{2} < y \leq \pi$,

$$y = \sec^{-1}(x) \text{ if and only if } \sec(y) = x. \quad (6.5.10)$$

For example, $\sec^{-1}(2) = \frac{\pi}{3}$, $\sec^{-1}(1) = 0$, $\sec^{-1}(-2) = \frac{2\pi}{3}$, and $\sec^{-1}(-1) = \pi$. Note that the domain of the arc secant function consists of the two intervals $(-\infty, -1]$ and $[1, \infty)$, the range of the secant function, and the range is composed of the two intervals $[0, \frac{\pi}{2})$ and $(\frac{\pi}{2}, \pi]$, the domain of the restricted secant function.

Since

$$\lim_{x \rightarrow \frac{\pi}{2}^-} \sec(x) = \infty$$

and

$$\lim_{x \rightarrow \frac{\pi}{2}^+} \sec(x) = -\infty,$$

it follows that

$$\lim_{x \rightarrow \infty} \sec^{-1}(x) = \frac{\pi}{2} \quad (6.5.11)$$

and

$$\lim_{x \rightarrow -\infty} \sec^{-1}(x) = \frac{\pi}{2}. \quad (6.5.12)$$

Thus the line $y = \frac{\pi}{2}$ is a horizontal asymptote for the graph of $y = \sec^{-1}(x)$ both as x goes to ∞ and as x goes to $-\infty$, as shown in Figure 6.5.3.

To find the derivative of the arc secant function, let $y = \sec^{-1}(x)$. Then $\sec(y) = x$, so

$$\frac{d}{dx} \sec(y) = \frac{d}{dx} x.$$

Hence

$$\sec(y) \tan(y) \frac{dy}{dx} = 1,$$

and so

$$\frac{dy}{dx} = \frac{1}{\sec(y) \tan(y)}.$$

Now $\sec(y) = x$ and

$$\tan^2(y) = \sec^2(y) - 1 = x^2 - 1.$$

Hence $\tan(y) = \pm\sqrt{x^2 - 1}$. If x is in $[1, \infty)$, then $0 \leq y < \frac{\pi}{2}$ and $\tan(y) \geq 0$; if x is in $(-\infty, -1]$, then $\frac{\pi}{2} < y \leq \pi$ and $\tan(y) \leq 0$. Thus

$$\sec(y) \tan(y) = \begin{cases} x\sqrt{x^2 - 1}, & \text{if } x \geq 1 \\ -x\sqrt{x^2 - 1}, & \text{if } x \leq -1. \end{cases}$$

Since $|x| = x$ when $x \geq 1$ and $|x| = -x$ when $x \leq -1$, it follows that

$$\sec(y) \tan(y) = |x|\sqrt{x^2 - 1}.$$

Hence

$$\frac{dy}{dx} = \frac{1}{|x|\sqrt{x^2 - 1}}.$$

Proposition

$$\frac{d}{dx} \sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2 - 1}}. \quad (6.5.13)$$

Example Using the chain rule, we have

$$\frac{d}{dx} \sec^{-1}(3x) = \frac{3}{|3x|\sqrt{9x^2 - 1}} = \frac{1}{|x|\sqrt{9x^2 - 1}}.$$

We will leave the definition of inverse functions for the cotangent, cosine, and cosecant functions for the problems at the end of the section. In the next section we will see how the arc tangent, arc sine, and arc secant functions are useful in evaluating certain integrals; the arc cotangent, arc cosine, and arc cosecant functions could be used in similar roles, but, wherever they are used, we could just as well use arc tangent, arc sine, or arc secant. Hence the former, although useful in other situations, will not be as important for our present study as the latter.

Problems

1. Find the derivatives of each of the following functions.

(a) $f(x) = x \tan^{-1}(x)$

(b) $g(t) = \tan^{-1}(3t^2)$

(c) $g(x) = \frac{\sin^{-1}(3x)}{x}$

(d) $f(x) = 3x \sec^{-1}(5x)$

2. Evaluate each of the following.

(a) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

(b) $\tan^{-1}(-\sqrt{3})$

(c) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

(d) $\sec^{-1}\left(-\frac{2}{\sqrt{3}}\right)$

(e) $\sin^{-1}\left(\sin\left(\frac{3\pi}{4}\right)\right)$

(f) $\sin\left(\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)\right)$

3. Evaluate the following integrals.

(a) $\int \frac{1}{1+2x^2} dx$

(b) $\int \frac{4x}{2+x^2} dx$

(c) $\int \frac{3}{x^2+4} dx$

(d) $\int \frac{5}{x^2+2x+3} dx$

(e) $\int \frac{x}{x^2+4x+5} dx$

(f) $\int \frac{x+1}{x^2+2x+6} dx$

(g) $\int_{-1}^1 \frac{1}{1+x^2} dx$

(h) $\int_0^{\frac{1}{3}} \frac{1}{1+9x^2} dx$

4. Evaluate the following integrals.

(a) $\int \frac{1}{x^3+x} dx$

(b) $\int \frac{2+x}{x(4x^2+1)} dx$

(c) $\int \frac{1}{x^2(x^2+1)} dx$

(d) $\int \frac{1}{(x+1)(x^2+2)} dx$

(e) $\int \sin^{-1}(x) dx$

(f) $\int \tan^{-1}(3x) dx$

(g) $\int \frac{5}{\sqrt{1-9x^2}} dx$

(h) $\int \frac{1}{\sqrt{4-8x^2}} dx$

(i) $\int \frac{3x}{\sqrt{1-x^2}} dx$

(j) $\int_{-2}^2 \frac{1}{\sqrt{16-x^2}} dx$

5. The cosine function has an inverse, called the *arc cosine function*, if its domain is restricted to $[0, \pi]$. That is, for $0 \leq y \leq \pi$,

$$y = \cos^{-1}(x) \text{ if and only if } \cos(y) = x.$$

(a) Show that $\frac{d}{dx} \cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$.

(b) Show that $\sec^{-1}(x) = \cos^{-1}\left(\frac{1}{x}\right)$.

(c) Use the result from (b) to find $\frac{d}{dx} \sec^{-1}(x)$.

(d) Use the fact that

$$\frac{d}{dx} \sin^{-1}(x) = \frac{d}{dx} (-\cos^{-1}(x))$$

to show that

$$\sin^{-1}(x) + \cos^{-1}(x) = \frac{\pi}{2}$$

for all x in $[-1, 1]$.

6. The cotangent function has an inverse, called the *arc cotangent function*, if its domain is restricted to $(0, \pi)$. That is, for $0 < y < \pi$,

$$y = \cot^{-1}(x) \text{ if and only if } \cot(y) = x.$$

Show that $\frac{d}{dx} \cot^{-1}(x) = -\frac{1}{1+x^2}$.

7. The cosecant function has an inverse, called the *arc cosecant function*, if its domain is restricted to the intervals $[-\frac{\pi}{2}, 0)$ and $(0, \frac{\pi}{2}]$. That is, for $-\frac{\pi}{2} \leq y < 0$ or $0 < y \leq \frac{\pi}{2}$,

$$y = \csc^{-1}(x) \text{ if and only if } \csc(y) = x.$$

Show that $\frac{d}{dx} \csc^{-1}(x) = -\frac{1}{|x|\sqrt{x^2-1}}$.

8. Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$.

9. (a) Use the fact that

$$\tan^{-1}(x) = \int_0^x \frac{1}{1+t^2} dt$$

to find the Taylor series expansion for $\tan^{-1}(x)$ about 0. On what interval does this series converge?

(b) Use your result in (a) and the fact that $\pi = 4 \tan^{-1}(1)$ to approximate π with an error of no more than 0.001.

10. (a) Show that

$$\frac{d}{dx} \tan^{-1}\left(\frac{1}{x}\right) = -\frac{1}{1+x^2}.$$

(b) Use the result from (a) to show that

$$\tan^{-1}(x) + \tan^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2}$$

for all $x > 0$.

(c) Find a result similar to (b) for $x < 0$.