Section 6.1

The Exponential Function

At this point we have seen all the major concepts of calculus: derivatives, integrals, and power series. For the rest of the book we will be concerned with how these ideas apply in various circumstances. In particular, in this chapter we will introduce the remaining elementary functions of calculus: the exponential function, the natural logarithm function, the inverse trigonometric functions, and the hyperbolic trigonometric functions. As they are introduced, we will discuss related issues involving derivatives, integrals, and power series, as well as applications to the physical world.

We will begin by considering the exponential function. We first saw this function in Section 5.7, but we will redefine it here for completeness.

Definition The exponential function, with value at x denoted by $\exp(x)$, is defined by

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots. \tag{6.1.1}$$

We saw in Section 5.7 that this series converges absolutely for all values of x; hence the domain of the exponential function is $(-\infty, \infty)$. We should also note that $\exp(0) = 1$.

Using the properties of power series, it is an easy matter to compute the derivative of the exponential function:

$$\frac{d}{dx}\exp(x) = \frac{d}{dx}\left(\sum_{n=0}^{\infty}\frac{x^n}{n!}\right) = \sum_{n=0}^{\infty}\frac{d}{dx}\left(\frac{x^n}{n!}\right) = \sum_{n=1}^{\infty}\frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty}\frac{x^n}{n!} = \exp(x).$$

Proposition

$$\frac{d}{dx}\exp(x) = \exp(x). \tag{6.1.2}$$

Example Using the chain rule, we have

$$\frac{d}{dx}\exp(4x) = 4\exp(4x).$$

Example Similarly,

$$\frac{d}{dx}\exp(x^2) = 2x\exp(x^2).$$

In fact, the exponential function is the only function f for which both $f(0) = 1$ and $f'(x) = f(x)$ for all x. To see this, we first demonstrate a more general property. Suppose

f is any function for which $f(0) = c$ and $f'(x) = kf(x)$ for all x, where c and k are constants. Then it follows that

$$f''(x) = \frac{d}{dx}(kf(x)) = kf'(x) = k^2 f(x),$$
$$f'''(x) = \frac{d}{dx}(k^2 f(x)) = k^2 f'(x) = k^3 f(x),$$

and, in general,

$$f^{(n)}(x) = k^n f(x) \tag{6.1.3}$$

for $n = 0, 1, 2, \ldots$. Hence

$$f^{(n)}(0) = k^n c \tag{6.1.4}$$

for $n = 0, 1, 2, \ldots$. Thus the Taylor series for f about 0 is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{ck^n}{n!} x^n = c \sum_{n=0}^{\infty} \frac{(kx)^n}{n!} = c \exp(kx), \tag{6.1.5}$$

where the final equality follows from the definition of the exponential function. Now, as a consequence of Taylor's theorem, if P_n is the nth order Taylor polynomial for f at 0, then

$$|f(x) - P_n(x)| \leq \frac{M}{(n+1)!}|x|^{n+1}, \tag{6.1.6}$$

where M is the maximum value of $|f^{(n+1)}|$ on the closed interval from 0 to x. But

$$f^{(n+1)}(x) = k^{n+1} f(x),$$

so

$$M = |k|^{n+1} L$$

where L is the maximum value of $|f|$ on the closed interval from 0 to x. Hence

$$|f(x) - P_n(x)| \leq \frac{|k|^{n+1} L}{(n+1)!}|x|^{n+1} = \frac{L|kx|^{n+1}}{(n+1)!}. \tag{6.1.7}$$

As we have seen before,

$$\lim_{n \to \infty} \frac{L|kx|^{n+1}}{(n+1)!} = 0 \tag{6.1.8}$$

for any value of x, so it follows that

$$f(x) = \lim_{n \to \infty} P_n(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \tag{6.1.9}$$

for all x. In other words, f has a Taylor series representation, and so, using (6.1.5), we have

$$f(x) = c \exp(x).$$

Proposition If f is a function for which $f(0) = c$ and $f'(x) = kf(x)$ for all x, where c and k are constants, then

$$f(x) = c\exp(x) \tag{6.1.10}$$

for all x.

In particular, if we let $c = 1$ and $k = 1$ in this proposition, then $f(x) = \exp(x)$. In many ways, it is this property that makes the exponential function one of the most important functions in mathematics.

Now consider a function f defined by $f(x) = \exp(x + b)$ for some constant b. Then

$$f'(x) = \exp(x + b) = f(x),$$

so, by the previous proposition, we must have $f(x) = c\exp(x)$ for all x, where $c = f(0) = \exp(b)$. That is, for all values of x,

$$\exp(x + b) = f(x) = \exp(b)\exp(x).$$

This demonstrates a fundamental algebraic property of the exponential function: For any numbers a and b,

$$\exp(a + b) = \exp(a)\exp(b). \tag{6.1.11}$$

It follows from (6.1.11) that for any number a,

$$\exp(a)\exp(-a) = \exp(a - a) = \exp(0) = 1.$$

That is,

$$exp(-a) = \frac{1}{\exp(a)} \tag{6.1.12}$$

More generally, using both (6.1.11) and (6.1.12), we have

$$\exp(a - b) = \exp(a)\exp(-b) = \frac{\exp(a)}{\exp(b)}. \tag{6.1.13}$$

for any numbers a and b, another important algebraic property of the exponential function.

We shall soon see that the number $\exp(1)$ plays a special role in this discussion.

Definition The value of the exponential function at 1 is denoted by e. That is,

$$e = \exp(1) = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \cdots. \tag{6.1.14}$$

It may be shown, although not easily, that e is an irrational number. Much more easily (see Problem 5), it may be shown that, to 5 decimal places, e is given by 2.71828. The use of the letter e to denote this number originates with Leonhard Euler (1707-1783), one of the most prolific mathematicians of all time.

Notice that for any positive integer n,

$$\exp(n) = \exp(\underbrace{1 + 1 + \cdots + 1}_{n \text{ times}}) = \underbrace{\exp(1)\exp(1)\cdots\exp(1)}_{n \text{ times}} = (\exp(1))^n = e^n \tag{6.1.15}$$

and

$$\exp(-n) = \frac{1}{\exp(n)} = \frac{1}{e^n} = e^{-n}. \tag{6.1.16}$$

Combining this with $\exp(0) = 1$, we have

$$\exp(n) = e^n \tag{6.1.17}$$

for all integers n. Moreover, for any integer $n \neq 0$,

$$\begin{aligned}\left(\exp\left(\frac{1}{n}\right)\right)^n &= \underbrace{\exp\left(\frac{1}{n}\right)\exp\left(\frac{1}{n}\right)\cdots\exp\left(\frac{1}{n}\right)}_{n \text{ times}} \\ &= \exp\underbrace{\left(\frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}\right)}_{n \text{ times}} \\ &= \exp(1) = e,\end{aligned}$$

showing that

$$\exp\left(\frac{1}{n}\right) = e^{\frac{1}{n}}. \tag{6.1.18}$$

Hence if m and n are integers with $n \neq 0$, then

$$\exp\left(\frac{m}{n}\right) = \exp\underbrace{\left(\frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}\right)}_{m \text{ times}} = \left(\exp\left(\frac{1}{n}\right)\right)^m = \left(e^{\frac{1}{n}}\right)^m = e^{\frac{m}{n}}. \tag{6.1.19}$$

The next proposition summarizes these facts.

Proposition For any rational number r,

$$\exp(r) = e^r. \tag{6.1.20}$$

That is, evaluating the exponential function at a rational number r is equivalent to raising e to the rth power. A natural question at this point is to ask whether the same result holds for irrational numbers. A little thought shows that this question is not meaningful; although we know what it means to raise a number to a rational power (namely, for integers m and n,

$$a^{\frac{m}{n}} = \sqrt[n]{a^m},$$

that is, $a^{\frac{m}{n}}$ is the nth root of the mth power of a), we have never defined what it means to raise a number to an irrational power. For example, at this point we do not have a meaning to associate with the symbol 2^π. We will now take the first step toward remedying this situation by defining e^s for an irrational number s.

Definition If s is an irrational number, then we define

$$e^s = \exp(s). \tag{6.1.21}$$

With this definition we can now say that

$$\exp(x) = e^x \tag{6.1.22}$$

for any real number x. The properties of the exponential function stated in (6.1.11) and (6.1.13) may be restated as

$$e^{x+y} = e^x e^y \tag{6.1.23}$$

and

$$e^{x-y} = \frac{e^x}{e^y} \tag{6.1.24}$$

for any real numbers x and y. Hence exponents behave in this new situation exactly the way we should expect them to behave.

From our previous result that

$$\frac{d}{dx}\exp(x) = \exp(x),$$

it now follows that

$$\frac{d}{dx}e^x = e^x. \tag{6.1.25}$$

From this differentiation rule we obtain the indefinite integral

$$\int e^x dx = e^x + c. \tag{6.1.26}$$

Example Using the chain rule, we have

$$\frac{d}{dx}e^{2x} = 2e^{2x}.$$

Example Using the product and chain rules,

$$\begin{aligned}\frac{d}{dx}\left(3xe^{4x^2}\right) &= 3x\frac{d}{dx}\left(e^{4x^2}\right) + e^{4x^2}\frac{d}{dx}(3x)\\ &= (3x)\left(8xe^{4x^2}\right) + \left(e^{4x^2}\right)(3)\\ &= (3+24x^2)e^{4x^2}.\end{aligned}$$

Example Since

$$\frac{d}{dx}e^{-4x} = -4e^{-4x},$$

it follows that

$$\int e^{-4x}dx = -\frac{1}{4}e^{-4x} + c.$$

Notice the similarity between the evaluation of the integral in the last example and the evaluation of the integral

$$\int \cos(-4x)dx = -\frac{1}{4}\sin(-4x) + c.$$

In fact, just as, for any $a \neq 0$,

$$\int \cos(ax)dx = \frac{1}{a}\sin(ax) + c,$$

we have

$$\int e^{ax}dx = \frac{1}{a}e^{ax} + c. \tag{6.1.27}$$

Example To evaluate $\int 3xe^{2x^2}dx$, we use the substitution

$$\begin{aligned} u &= 2x^2 \\ du &= 4xdx. \end{aligned}$$

Then $\frac{1}{4}du = xdx$, so

$$\int 3xe^{2x^2}dx = \frac{3}{4}\int e^u du = \frac{3}{4}e^u + c = \frac{3}{4}e^{2x^2} + c.$$

Example To evaluate $\int 2xe^x dx$, we use integration by parts with

$$\begin{aligned} u &= 2x & dv &= e^x dx \\ du &= 2dx & v &= e^x. \end{aligned}$$

Then

$$\int 2xe^x dx = 2xe^x - \int 2e^x dx = 2xe^x - 2e^x + c.$$

Notice the similarity between the technique for evaluating the integral in the last example and the technique for evaluating $\int 2x\sin(x)dx$.

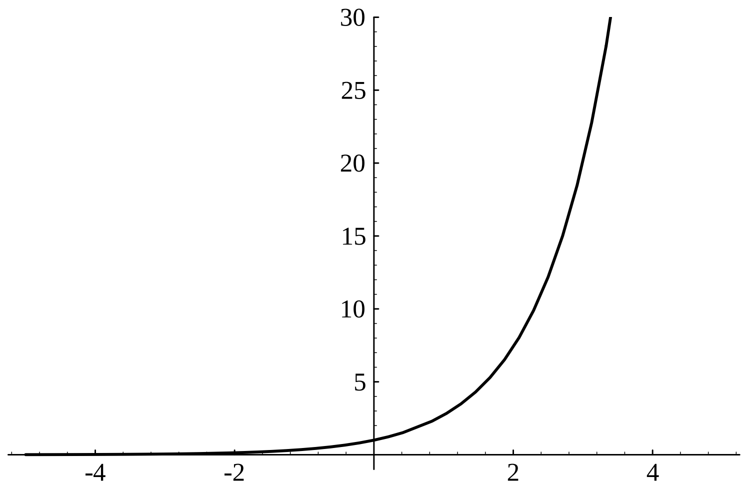

Figure 6.1.1 Graph of $y = e^x$

Example The integral $\int e^x \sin(x)dx$ may also be handled by integration by parts, although with a little more work than in the previous example. Here we will let

$$\begin{aligned} u &= \sin(x) & dv &= e^x dx \\ du &= \cos(x)dx & v &= e^x. \end{aligned}$$

Then

$$\int e^x \sin(x)dx = e^x \sin(x) - \int e^x \cos(x)dx.$$

We now perform another integration by parts by choosing

$$\begin{aligned} u &= \cos(x) & dv &= e^x dx \\ du &= -\sin(x)dx & v &= e^x. \end{aligned}$$

Then

$$\begin{aligned} \int e^x \sin(x) &= e^x \sin(x) - \left(e^x \cos(x) + \int e^x \sin(x)dx\right) \\ &= e^x \sin(x) - e^x \cos(x) - \int e^x \sin(x)dx. \end{aligned}$$

At first glance it may seem that we are back to where we started; however, all we need to do now is solve for $\int e^x \sin(x)dx$. That is, we have

$$2\int e^x \sin(x)dx = e^x \sin(x) - e^x \cos(x) = e^x(\sin(x) - \cos(x)),$$

so

$$\int e^x \sin(x)dx = \frac{1}{2}e^x(\sin(x) - \cos(x)) + c.$$

Note that we have added an arbitrary constant c since we are seeking the general antiderivative.

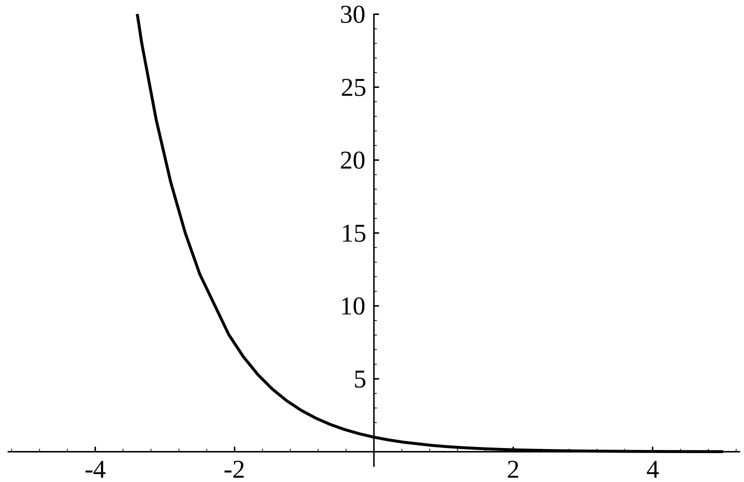

Figure 6.1.2 Graph of $y = e^{-x}$

We now have sufficient information about the exponential function to understand the geometry of its graph. Since $e > 0$, we know that $e^x > 0$ for all rational values of x, and hence, by continuity, for all values of x. Since $e > 1$, it follows that

$$\lim_{x \to \infty} e^x = \infty \tag{6.1.28}$$

and

$$\lim_{x \to -\infty} e^x = \lim_{u \to \infty} e^{-u} = \lim_{u \to \infty} \frac{1}{e^u} = 0. \tag{6.1.29}$$

Moreover, since

$$\frac{d}{dx} e^x = e^x > 0 \tag{6.1.30}$$

and

$$\frac{d^2}{dx^2} e^x = e^x > 0 \tag{6.1.31}$$

for all x, the graph of $y = e^x$ is always increasing and always concave up. Moreover, (6.1.30) and (6.1.31) indicate that as x increases, the graph is not only increasing, but its slope is increasing at the same rate that y is increasing. Thus we should expect y to grow at a very rapid rate, as we see in Figure 6.1.1. This rate of growth is characterized as *exponential growth*. Figure 6.1.2 shows the graph of $y = e^{-x}$, which is the graph of $y = e^x$ reflected about the y-axis. In this case y decreases asymptotically toward 0 as x increases; this is known as *exponential decay*

We will close this section with an application to the problem of uninhibited population growth, a problem we first considered in Section 1.4.

Uninhibited population growth

Recall from Section 1.4 that if x_n represents the size of a population after n units of time and the population grows at a constant rate of $\alpha 100\%$ per unit of time, then the sequence $\{x_n\}$ must satisfy the linear difference equation

$$x_{n+1} - x_n = \alpha x_n \tag{6.1.32}$$

for $n = 0, 1, 2, \ldots$. At that time we saw that the solution of this equation is given by

$$x_n = (1+\alpha)^n x_0. \tag{6.1.33}$$

The crucial aspect of (6.1.32) is the statement that amount of change in the size of the population over any unit of time is proportional to the current size of the population. Hence if $x(t)$ represents the size of a population at time t, where the population can change continuously over time, then the continuous time analog of (6.1.32) is the *differential equation*

$$\dot{x}(t) = \alpha x(t) \tag{6.1.34}$$

for all time t. If x_0 is the size of the population at time $t = 0$, then we know from our work in this section that the only solution to this equation is the function

$$x(t) = x_0 e^{\alpha t}. \tag{6.1.35}$$

Hence if the size of a population is growing at a rate which is proportional to itself, an assumption which, as we noted in Section 1.4, is often reasonable over short periods of time, then the population will grow exponentially. As in Section 1.4, we refer to such growth as *uninhibited population growth*.

Example In 1970 the population of the United States was 203.3 million and in 1980 the population was 226.5 million. Assuming an uninhibited growth model and letting $x(t)$ represent the population t years after 1970, by (6.1.35) we should have

$$x(t) = 203.3e^{\alpha t}$$

for some constant α. Since $x(10) = 226.5$, we can find α by solving

$$226.5 = 203.3e^{10\alpha}.$$

That is, we need to find a value for α such that

$$e^{10\alpha} = \frac{226.5}{203.3} = 1.114.$$

Unfortunately, solving this equation exactly requires being able to reverse the process of applying the exponential function. In other words, we need an inverse for the exponential function. We shall take up that problem in the next section; for now we may use a numerical approximation. You should verify that $\alpha = 0.0108$ satisfies the equation. Thus this model would predict the population of the United States t years after 1970 to be

$$x(t) = 203.3e^{0.0108t}.$$

For example, this model would predict a 1990 population of

$$x(20) = 203.3e^{(0.0108)(20)} = 252.3$$

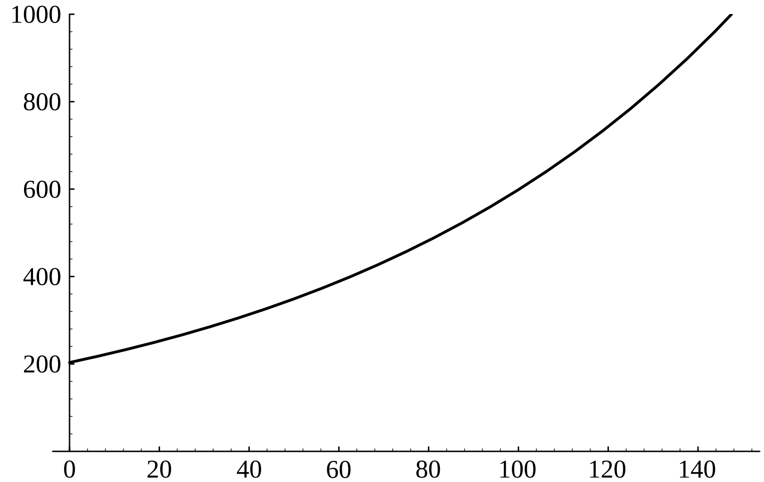

Figure 6.1.3 Uninhibited growth model for the United States (1970-2120)

and a population in the year 2000 of

$$x(30) = 203.3e^{(0.0108)(30)} = 281.1.$$

While the prediction for 1990 is fairly accurate (the actual population was approximately 249.6 million), the second prediction differs significantly from the Census Bureau's own prediction of a population of 268.3 million for the year 2000. As we discussed in Sections 1.4 and 1.5, an uninhibited growth model is a simple model which cannot be expected to be accurate for predictions too far into the future.

We shall have more to say about population models in Section 6.3, where we will also see another example of a differential equation. We will have a much fuller discussion of differential equations in Chapter 8.

Problems

1. Find the derivative of each of the following functions.

   (a) $f(x) = 3e^{2x}$ (b) $g(t) = 4t^2e^{3t}$

   (c) $h(z) = (3z^2 - 6)e^{5z^3}$ (d) $f(x) = e^{3x}\sin(2x)$

   (e) $g(x) = \dfrac{3x}{2e^x}$ (f) $h(t) = e^{-6t}\cos(4t)$

   (g) $f(s) = \dfrac{3s-1}{e^{-2s}+2}$ (h) $g(\theta) = 5\theta e^{6\theta}\sin(2\theta)$

2. Evaluate each of the following integrals.

   (a) $\displaystyle\int 3e^{2x}dx$ (b) $\displaystyle\int 4xe^{3x^2}dx$

   (c) $\displaystyle\int 4te^{3t}dt$ (d) $\displaystyle\int 5ye^{-y}dy$

(e) $\displaystyle\int z^2 e^z dz$

(f) $\displaystyle\int x^3 e^{-2x} dx$

(g) $\displaystyle\int e^x \cos(x) dx$

(h) $\displaystyle\int e^{-2t} \sin(3t) dt$

3. Find the maximum value of $f(x) = x^2 e^{-x}$ on the interval $(0, \infty)$.

4. (a) Use the Taylor series for e^{-x} to show that $e^{-1} > \frac{1}{3}$. Hence conclude that $e < 3$.
   (b) Show that if P_n is the nth order Taylor polynomial for e^x at 0, then

$$|e^x - P_n(x)| \leq \frac{3^x}{(n+1)!}|x|^{n+1}$$

   for any value of x.
   (c) Use (b) to find an approximation for e with an error of less than 0.000005.

5. Find the following limits.

(a) $\displaystyle\lim_{x \to \infty} xe^{-x}$

(b) $\displaystyle\lim_{x \to 0} \frac{e^x - 1}{x}$

(c) $\displaystyle\lim_{t \to 0} \frac{e^{-t} - 1}{t}$

(d) $\displaystyle\lim_{x \to \infty} x^2 e^{-2x}$

6. (a) Show that $\displaystyle\lim_{x \to \infty} x^n e^{-x} = 0$ for any positive integer n.
   (b) Use (a) to show that if p is any polynomial, then $\displaystyle\lim_{x \to \infty} p(x)e^{-x} = 0$. This shows that e^x grows faster as $x \to \infty$ than any polynomial function.

7. Graph the following functions on the specified intervals.

(a) $f(t) = 3e^{2t}$ on $[-3, 3]$

(b) $g(x) = 4x^2 e^{-2x}$ on $[0, 5]$

(c) $g(t) = \dfrac{e^t - e^{-t}}{e^t + e^{-t}}$ on $[-10, 10]$

(d) $f(t) = e^{-\frac{t}{4}} \sin(3t)$ on $[0, 10]$

8. Evaluate the following improper integrals.

(a) $\displaystyle\int_0^\infty e^{-x} dx$

(b) $\displaystyle\int_0^\infty 3e^{-2x} dx$

(c) $\displaystyle\int_0^\infty xe^{-x} dx$

(d) $\displaystyle\int_0^\infty 3xe^{-2x} dx$

(e) $\displaystyle\int_0^\infty x^2 e^{-x} x dx$

(f) $\displaystyle\int_0^\infty xe^{-x^2} dx$

9. Use the integral test to show that the infinite series $\displaystyle\sum_{n=0}^{\infty} e^{-n}$ converges.

10. (a) Find the Taylor series for $f(x) = e^{-x^2}$.
    (b) Use (a) to find the the Taylor series for

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt,$$

known as the *error function*.

(c) Use your result in (b) to approximate erf(1) with an error less than 0.0001.

11. Suppose $x(t)$ is the population of a certain country t years after 1985, $x(0) = 23.4$ million, and

$$\dot{x}(t) = 0.008x(t).$$

(a) What will the population of the country be in the year 2000?

(b) In what year will the population be twice what it was in 1985?

12. Let

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

(a) Graph f on the interval $[-5, 5]$.

(b) Show that $f'(0) = 0$.

(c) Show that $f^{(n)}(0) = 0$ for $n = 0, 1, 2, \ldots$.

(d) Show that f is C^∞ on $(-\infty, \infty)$.

(e) Note that the Taylor series for f about 0 converges for all x in $(-\infty, \infty)$, but does not converge to $f(x)$ except at 0. Thus f is C^∞ on $(-\infty, \infty)$, but not analytic at 0.