Difference Equations
to
Differential Equations

Section 5.8

Taylor Series

In this section we will put together much of the work of Sections 5.1-5.7 in the context of a discussion of Taylor series. We begin with two definitions.

Definition If f is a function such that $f^{(n)}$ is continuous on an open interval (a, b) for $n = 0, 1, 2, \ldots$, then we say f is C^∞ on (a, b).

Definition If f is C^∞ on an interval (a, b) and c is a point in (a, b), then the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots \quad (5.8.1)$$

is called the *Taylor series* for f about c.

A Taylor series is a power series constructed from a given function in the same manner as a Taylor polynomial. As with any power series about c, the Taylor series for a function f about c converges at $x = c$, but does not necessarily converge at any other points. If it does converge for other values of x, it will converge absolutely on an interval $(c-R, c+R)$, where R is the radius of convergence. However, even if the series converges at $x \neq c$, it need not converge to $f(x)$. That is, a function may be C^∞ without being analytic. (See Problem 12 of Section 6.1 for an example.) If the Taylor series does converge to $f(x)$ for all x in the interval of convergence, then it is the unique power series representation for f on this interval.

If P_n is the nth order Taylor polynomial for f at c, then P_n is a partial sum of the Taylor series for f about c. Hence to show that the Taylor series converges to f at x, we need to show that

$$f(x) = \lim_{n\to\infty} P_n(x). \quad (5.8.2)$$

Equivalently, we need to show that

$$\lim_{n\to\infty} r_n(x) = 0, \quad (5.8.3)$$

where

$$r_n(x) = f(x) - P_n(x). \quad (5.8.4)$$

In this regard, the error bounds for $r_n(x)$ developed in Section 5.2 can be very useful.

Example For any $n = 0, 1, 2, \ldots$, if P_{2n+1} is the Taylor polynomial of order $2n+1$ for $f(x) = \sin(x)$ at 0, then

$$P_{2n+1}(x) = \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!}.$$

In Section 5.2 we saw that if

$$r_{2n+1}(x) = \sin(x) - P_{2n+1}(x),$$

then

$$|r_{2n+1}(x)| \leq \frac{|x|^{2n+3}}{(2n+3)!}$$

for any value of x. In Section 5.7 we saw that, for any x,

$$\lim_{n\to\infty} \frac{|x|^{2n+3}}{(2n+3)!} = 0,$$

so

$$\lim_{n\to\infty} |r_{2n+1}(x)| = 0.$$

Hence

$$\sin(x) = \lim_{n\to\infty} P_{2n+1}(x)$$

for all x. That is,

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \tag{5.8.5}$$

for all x. Thus the Taylor series for $\sin(x)$ about 0 provides a power series representation for $\sin(x)$ on the interval $(-\infty, \infty)$. Note that this example is essentially a restatement of our second example in Section 5.7.

In many cases showing

$$\lim_{n\to\infty} r_n(x) = 0 \tag{5.8.6}$$

is difficult. However, since power series representations are unique, if we are able to find a power series representation for a given function by manipulating some other known representation, then we know that this series is the Taylor series for that function. This is in fact the way many Taylor series representations are found in practice.

Example Since

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

for $-1 < x < 1$, it follows that

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \cdots$$

for $-1 < -x < 1$, that is, $-1 < x < 1$. Hence we have found a Taylor series representation for

$$f(x) = \frac{1}{1+x}$$

on $(-1, 1)$.

Example Similar to the previous example, we have

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + \cdots$$

for $-1 < x^2 < 1$, that is, $-1 < x < 1$. Thus we have found a Taylor series representation for

$$f(x) = \frac{1}{1+x^2}$$

on $(-1, 1)$.

Example In Section 5.7 we saw how the relationship

$$\cos(x) = 1 - \int_0^x \sin(t)dt$$

combined with the Taylor series representation

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

yields

$$\cos(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \tag{5.8.7}$$

for all values of x. Thus (5.8.7) is the Taylor series representation for $\cos(x)$ about 0 on $(-\infty, \infty)$.

Example Since

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

for all values of x, it follows that

$$\frac{\sin(x)}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

for all $x \neq 0$. In fact, if we define

$$f(x) = \begin{cases} \dfrac{\sin(x)}{x}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0, \end{cases}$$

then the Taylor series representation for f about 0 on $(-\infty, \infty)$ is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots \tag{5.8.8}$$

Example Since

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

for $-1 < x < 1$,

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} \frac{d}{dx} x^n = \sum_{n=1}^{\infty} n x^{n-1}$$

for $-1 < x < 1$. But

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2},$$

so we have the Taylor series representation

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

for all x in $(-1, 1)$.

The final two examples of this section will illustrate the use of Taylor series in solving problems that we could not handle before.

Example Define

$$f(x) = \begin{cases} \dfrac{\sin(x)}{x}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases}$$

Then, as we saw above,

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

is the Taylor series representation for f about 0 on $(-\infty, \infty)$. Now f is continuous on $(-\infty, \infty)$ and so has an antiderivative on $(-\infty, \infty)$, but, as we have mentioned before, this antiderivative is not expressible in terms of the elementary functions of calculus. However, by the Fundamental Theorem of Calculus, the function

$$\mathrm{Si}(x) = \int_0^x f(t)dt, \tag{5.8.9}$$

called the *sine integral function*, is an antiderivative of f. Moreover, even though we cannot express this integral in terms of the elementary functions, we can find its Taylor series representation. That is,

$$\begin{aligned}
\mathrm{Si}(x) &= \int_0^x \left(\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n+1)!} \right) dt \\
&= \sum_{n=0}^{\infty} \int_0^x \frac{(-1)^n t^{2n}}{(2n+1)!} \, dt \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)(2n+1)!} \bigg|_0^x \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!} \\
&= x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \frac{x^7}{7 \cdot 7!} + \cdots \qquad (5.8.10)
\end{aligned}$$

for all values of x. In particular,

$$\mathrm{Si}(1) = \int_0^1 \frac{\sin(x)}{x} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!} = 1 - \frac{1}{3 \cdot 3!} + \frac{1}{5 \cdot 5!} - \frac{1}{7 \cdot 7!} + \cdots.$$

Since this is an alternating series which satisfies the conditions of Leibniz's theorem, if

$$s_n = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)(2k+1)!},$$

then

$$|\mathrm{Si}(1) - s_n| \leq \frac{1}{(2n+3)(2n+3)!}.$$

For example, if we want to approximate Si(1) with an error of no more than 0.0001, we note that for $n = 1$ we have, to 6 decimal places,

$$\frac{1}{(2n+3)(2n+3)!} = \frac{1}{5 \cdot 5!} = \frac{1}{600} = 0.001667,$$

while for $n = 2$ we have

$$\frac{1}{(2n+3)(2n+3)!} = \frac{1}{7 \cdot 7!} = \frac{1}{35,280} = 0.000028.$$

Thus

$$s_2 = 1 - \frac{1}{3 \cdot 3!} + \frac{1}{5 \cdot 5!} = 0.946111$$

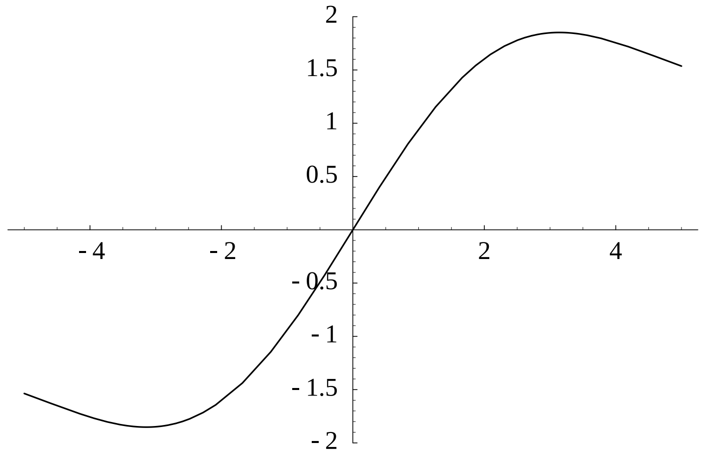

Figure 5.8.1 Taylor polynomial approximation to the graph of $y = \mathrm{Si}(x)$

differs from Si(1) by no more than 0.000028. In fact, since the next term in the series is negative, Si(1) must lie between 0.946111 and

$$0.946111 - 0.000028 = .946083.$$

In particular, we know that

$$\mathrm{Si}(1) = 0.9461$$

to 4 decimal places. Of course, this particular result could also be obtained using numerical integration. However, the point is that (5.8.10) gives us much more; it not only gives us an easy method to evaluate $\mathrm{Si}(x)$ for any value of x to any desired level of accuracy, but it also gives us an algebraic representation of the sine integral function which can be used in applications in much the same way that polynomials are used. In Figure 5.8.1 we have used the Taylor polynomial

$$P_{11}(x) = x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \frac{x^7}{7 \cdot 7!} + \frac{x^9}{9 \cdot 9!} - \frac{x^{11}}{11 \cdot 11!}$$

to approximate the graph of $\mathrm{Si}(x)$ on the interval $[-5, 5]$. Note that on this interval

$$|\mathrm{Si}(x) - P_{11}(x)| \leq \frac{5^{13}}{13 \cdot 13!} = 0.0151$$

to 4 decimal places, certainly accurate enough for the purposes of our graph.

Example Using

$$\frac{1}{x} = \frac{1}{1 - (1 - x)}$$

and

$$\frac{1}{1-x} = \sum_{n-0}^{\infty} x^n$$

for $-1 < x < 1$, we have

$$\frac{1}{x} = \sum_{n=0}^{\infty}(1-x)^n = \sum_{n=0}^{\infty}(-1)^n(x-1)^n \tag{5.8.11}$$

for $-1 < 1-x < 1$, that is, $0 < x < 2$. Hence (5.8.11) gives the Taylor series representation for

$$f(x) = \frac{1}{x}$$

about 1. Similar to our work in the previous example, we may now find an antiderivative for f on $(0,2)$ by integration. Namely,

$$\begin{aligned}
\int_1^x \frac{1}{t}\,dt &= \int_1^x \left(\sum_{n=0}^{\infty}(-1)^n(t-1)^n\right)dt \\
&= \sum_{n=0}^{\infty}\int_1^x (-1)^n(t-1)^n dt \\
&= \sum_{n=0}^{\infty}\left.\frac{(-1)^n(t-1)^{n+1}}{n+1}\right|_1^x \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n(x-1)^{n+1}}{n+1} \\
&= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots
\end{aligned}$$

provides a Taylor series representation for an antiderivative of f on the interval $(0,2)$. In Chapter 6 we will call this function the *natural logarithm function*, denoted $\log(x)$, although there we will use other means in order to define it on the interval $(0,\infty)$. In particular, note that this series converges at $x = 2$ as well, giving us, with this definition of $\log(x)$,

$$\log(2) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}.$$

Hence $\log(2)$ is the sum of the alternating harmonic series, a number for which we found an approximation in Section 5.6.

Problems

1. Show directly that

$$\cos(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$

for all x in $(-\infty,\infty)$.

2. Using any method, find Taylor series representations about 0 for the following functions. State the interval on which the representation is valid. Also, write out the first five nonzero terms of each series.
   (a) $\cos(x^2)$
   (b) $\sin(2x)$
   (c) $\dfrac{1}{1-t^2}$
   (d) $\dfrac{1}{2x-1}$
   (e) $\dfrac{1}{(1+t)^2}$
   (f) $\dfrac{1}{1+4x^2}$
   (g) $f(x) = \begin{cases} \dfrac{1-\cos(x)}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0 \end{cases}$
3. (a) Use the identity
$$\cos^2(x) = \frac{1+\cos(2x)}{2}$$
   to find the Taylor series representation for $\cos^2(x)$ about 0. On what interval is this representation valid?
   (b) What is the Taylor polynomial of order 8 for $\cos^2(x)$ at 0?
4. (a) Use Problem 3 and the identity
$$\sin^2(x) = 1 - \cos^2(x)$$
   to find the Taylor series representation for $\sin^2(x)$ about 0. On what interval is this representation valid?
   (b) What is the Taylor polynomial of order 8 for $\sin^2(x)$ at 0?
5. (a) Use the Taylor series representation about 0 for $\sin(x)$ to find the Taylor series representation for $\sin(x^2)$ about 0. On what interval is this representation valid?
   (b) What is the Taylor polynomial of order 10 for $\sin(x^2)$ at 0?
   (c) Find the Taylor series representation about 0 for
$$S(x) = \int_0^x \sin(t^2)dt.$$
   On what interval is this representation valid?
   (d) What is the Taylor polynomial of order 11 for $S(x)$ at 0?
   (e) Approximate $S(1)$ with an error of less than 0.00001.
6. Let P_n be the Taylor polynomial of order n at 0 for
$$f(x) = \frac{1}{1+x^2}.$$
   Plot f, P_2, P_4, and P_{10} together over the interval $[-1.5, 1.5]$. Why do the Taylor polynomials not give a good approximation to $f(x)$ when $|x| > 1$?
7. Find $\left.\dfrac{d^9}{dx^9}\mathrm{Si}(x)\right|_{x=0}$.