

***Difference Equations
to
Differential Equations***

**Section 5.7
Power Series**

We are now in a position to pick up the story we left off in Section 5.2: the extension of Taylor polynomials to Taylor series. We shall see that a Taylor series is a type of infinite series whose n th partial sum is a Taylor polynomial. Such series are examples of power series, objects that we will study in this section before considering Taylor series in Section 5.8.

Definition An infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots \quad (5.7.1)$$

is called a *power series* in x about c .

Example The infinite series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

is a power series in x about 0. Note that if we let

$$b_n = \left| \frac{x^n}{n!} \right| = \frac{|x|^n}{n!}$$

for $n = 0, 1, 2, \dots$, then

$$\lim_{n \rightarrow \infty} \frac{b_{n+1}}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{|x|^{n+1}}{(n+1)!}}{\frac{|x|^n}{n!}} = \lim_{n \rightarrow \infty} \frac{|x|}{n+1} = 0$$

for any value of x . That is, by the ratio test, the series is absolutely convergent, and hence convergent, for any value of x . Thus if we define a function, called the *exponential function*, by

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (5.7.2)$$

then this function is defined for all values of x . We shall have much more to say about this function, which may be thought of as the simplest “infinite” polynomial which is defined for all real numbers, in Chapter 6.

Notice that the convergence of (5.7.2) for all x implies, by the n th term test for divergence, that

$$\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0 \quad (5.7.3)$$

for any value of x . We have seen particular cases of this limit in the past, but this is the first time we have had a simple proof that it is always 0.

Example Recall that the Taylor polynomial of order $2n + 1$ for $\sin(x)$ at 0 is

$$P_{2n+1}(x) = \sum_{k=0}^n \frac{(-1)^k x^{2k+1}}{(2k+1)!}.$$

Hence $P_{2n+1}(x)$ is a partial sum of the power series

$$\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}. \quad (5.7.4)$$

If, for $k = 0, 1, 2, \dots$, we let

$$b_k = \left| \frac{(-1)^k x^{2k+1}}{(2k+1)!} \right| = \frac{|x|^{2k+1}}{(2k+1)!},$$

then

$$\lim_{k \rightarrow \infty} \frac{b_{k+1}}{b_k} = \lim_{k \rightarrow \infty} \frac{\frac{|x|^{2k+3}}{(2k+3)!}}{\frac{|x|^{2k+1}}{(2k+1)!}} = \lim_{k \rightarrow \infty} \frac{|x|^2}{(2k+3)(2k+2)} = 0$$

for all values of x . Thus, by the ratio test, (5.7.4) is absolutely convergent, and hence convergent, for all values of x . Moreover, from our work in Section 5.2, we know that

$$|\sin(x) - P_{2n+1}(x)| \leq \frac{|x|^{2n+3}}{(2n+3)!}$$

for all values of x . Since (using (5.7.3))

$$\lim_{n \rightarrow \infty} \frac{|x|^{2n+3}}{(2n+3)!} = 0,$$

it follows that

$$\lim_{n \rightarrow \infty} |\sin(x) - P_{2n+1}(x)| = 0$$

for all values of x . Hence

$$\sin(x) = \lim_{n \rightarrow \infty} P_{2n+1}(x)$$

for all values of x . That is, for any value of x ,

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}. \quad (5.7.5)$$

Example The series $\sum_{n=0}^{\infty} x^n$ is a power series in x about 0. From our work on geometric series, we know that this series will converge absolutely when $-1 < x < 1$ and will diverge otherwise. In fact, we have seen that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

for $-1 < x < 1$.

These examples show that some functions may be expressed as power series. Such functions are examples of *analytic functions*, which we now define.

Definition If f is a function for which there exists constants a_0, a_1, a_2, \dots such that

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n \quad (5.7.6)$$

for all values of x in some open interval about c , then we say f is *analytic at c* . If for some $h > 0$ the equality (5.7.6) holds for all x in the interval $I = (c-h, c+h)$, then we say f is *analytic on I* and we call

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

a *power series representation* of f on I .

Example From the previous examples we see that

$$f(x) = \exp(x)$$

and

$$g(x) = \sin(x)$$

are analytic on $(-\infty, \infty)$ and

$$h(x) = \frac{1}{1-x}$$

is analytic on $(-1, 1)$.

Before we can work effectively with power series we need to consider their convergence behavior. First note that the power series

$$\sum_{n=0}^{\infty} a_n (x-c)^n \quad (5.7.7)$$

converges at $x = c$ since in that case all terms after the first are 0. Next, suppose the series converges at a point $x = c + r$, where $r > 0$. That is, suppose $\sum_{n=0}^{\infty} a_n r^n$ converges. Then, by the n th term test for divergence,

$$\lim_{n \rightarrow \infty} a_n r^n = 0. \quad (5.7.8)$$

In particular, there exists an integer N such that

$$|a_n r^n| < 1 \quad (5.7.9)$$

for all $n > N$. Hence for any x and $n > N$,

$$\frac{|a_n (x - c)^n|}{\left| \frac{x - c}{r} \right|^n} = |a_n r^n| < 1. \quad (5.7.10)$$

In particular, $|a_n (x - c)^n|$ is

$$O\left(\left|\frac{x - c}{r}\right|^n\right).$$

Now if $c - r < x < c + r$, then

$$\left|\frac{x - c}{r}\right| < 1$$

and

$$\sum_{n=0}^{\infty} \left|\frac{x - c}{r}\right|^n$$

is a convergent geometric series. Thus, by the limit comparison test,

$$\sum_{n=0}^{\infty} a_n (x - c)^n$$

converges absolutely. In other words, we have shown that if (5.7.7) converges at $c + r$ with $r > 0$, then it converges absolutely for all x in $(c - r, c + r)$. The same argument works to show that if (5.7.7) converges at $c - r$ with $r > 0$, then it converges absolutely for all x in $(c - r, c + r)$. Letting R be the largest real number such that (5.7.7) converges absolutely for all x for which $|x - c| < R$, where we allow $R = \infty$ if (5.7.7) converges for all x , it follows that

$$\sum_{n=0}^{\infty} a_n (x - c)^n$$

converges absolutely on $(c - R, c + R)$ ($(-\infty, \infty)$ if $R = \infty$) and diverges for all x with $|x - c| > R$.

Proposition For a power series

$$\sum_{n=0}^{\infty} a_n(x - c)^n,$$

there exists an R , with $R = 0$, $R > 0$, or $R = \infty$, such that the series converges absolutely for all x satisfying $|x - c| < R$ and diverges for all x satisfying $|x - c| > R$.

Definition With the notation of the previous proposition, the interval $(c - R, c + R)$ ($(-\infty, \infty)$ if $R = \infty$) is called the *interval of convergence* and R is called the *radius of convergence* of the power series.

Note that the proposition does not say anything about the behavior of the series at $x = c - R$ or $x = c + R$. In fact, any type of behavior is possible at the endpoints of the interval of convergence; for a given series, these points must be checked individually for convergence. Moreover, although the proposition does not provide a method for finding the interval of convergence of a series, the next examples illustrate that the ratio test is very useful in this regard.

Example Consider the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n2^n}. \quad (5.7.11)$$

If we let

$$b_n = \left| \frac{(-1)^{n+1} x^n}{n2^n} \right| = \frac{|x|^n}{n2^n},$$

then

$$\lim_{n \rightarrow \infty} \frac{b_{n+1}}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{|x|^{n+1}}{(n+1)2^{n+1}}}{\frac{|x|^n}{n2^n}} = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right) \frac{|x|}{2} = \frac{|x|}{2}.$$

Hence, by the ratio test, (5.7.11) is absolutely convergent when

$$\frac{|x|}{2} < 1,$$

that is, when $-2 < x < 2$. Thus the radius of convergence is $R = 2$ and the interval of convergence is $(-2, 2)$. Now at $x = -2$, (5.7.11) becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} (-2)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (-1)^n 2^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{n} = \sum_{n=1}^{\infty} -\frac{1}{n},$$

which is a multiple of the harmonic series and hence divergent. At $x = 2$, (5.7.11) becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n},$$

which is the alternating harmonic series and hence convergent, although not absolutely convergent. Putting this together, we see that the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n 2^n}$$

converges absolutely for all x in $(-2, 2)$, converges conditionally at $x = 2$, and diverges for all other x .

Example In the first example of this section we used the ratio test to show that the power series

$$\sum_{n=1}^{\infty} \frac{x^n}{n!}$$

converges absolutely for all values of x . Hence in this case the interval of convergence is $(-\infty, \infty)$ and the radius of convergence is $R = \infty$.

Example Consider the power series $\sum_{n=1}^{\infty} n! x^n$. If we let

$$b_n = |n! x^n| = n! |x|^n,$$

then

$$\lim_{n \rightarrow \infty} \frac{b_{n+1}}{b_n} = \lim_{n \rightarrow \infty} \frac{(n+1)! |x|^{n+1}}{n! |x|^n} = \lim_{n \rightarrow \infty} (n+1) |x| = \begin{cases} 0, & \text{if } x = 0, \\ \infty, & \text{if } x \neq 0. \end{cases}$$

Hence, by the ratio test, this power series converges only when $x = 0$. Accordingly, the radius of convergence is $R = 0$.

Example Consider the power series

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n}. \tag{5.7.12}$$

If we let

$$b_n = \left| \frac{(x-1)^n}{n} \right| = \frac{|x-1|^n}{n},$$

then

$$\lim_{n \rightarrow \infty} \frac{b_{n+1}}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{|x-1|^{n+1}}{n+1}}{\frac{|x-1|^n}{n}} = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right) |x-1| = |x-1|.$$

Using the ratio test, we see that (5.7.12) converges absolutely when $|x-1| < 1$. Thus the radius of convergence is $R = 1$ and the interval of convergence is $(0, 2)$. At $x = 0$, (5.7.12) becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n},$$

which is a multiple of the alternating harmonic series and so converges conditionally. At $x = 2$, (5.7.12) becomes

$$\sum_{n=1}^{\infty} \frac{1}{n},$$

which is the harmonic series and so diverges. Hence the power series

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n}$$

converges absolutely for x in the interval $(0, 2)$, converges conditionally at $x = 0$, and diverges for all other x .

A power series resembles a polynomial; in fact, often it is convenient to think of a power series as a polynomial of infinite degree. Among the many nice properties of polynomials is the ease with which they may be differentiated and integrated. Our next result states that power series may be differentiated and integrated term by term in the same manner as polynomials. Although we have the tools to provide justifications for these statements, they are technical and perhaps best left to a more advanced text.

Differentiation and integration of power series Suppose the radius of convergence of the power series

$$\sum_{n=0}^{\infty} a_n(x-c)^n$$

is $R > 0$ and let

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$$

for x in $(c-R, c+R)$. Then

$$f'(x) = \sum_{n=0}^{\infty} \frac{d}{dx} a_n(x-c)^n = \sum_{n=1}^{\infty} n a_n(x-c)^{n-1} \quad (5.7.13)$$

for all x in $(c-R, c+R)$ and

$$\int_c^b f(x) dx = \sum_{n=0}^{\infty} \left(\int_c^b a_n(x-c)^n dx \right) = \sum_{n=0}^{\infty} \frac{a_n(b-c)^{n+1}}{n+1} \quad (5.7.14)$$

for all b in $(c-R, c+R)$.

Example Recall that the interval of convergence of

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

is $(-\infty, \infty)$. Hence

$$\begin{aligned} \frac{d}{dx} \exp(x) &= \frac{d}{dx} \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{d}{dx} \left(\frac{x^n}{n!} \right) \\ &= \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} \\ &= \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ &= \exp(x) \end{aligned}$$

for all x in $(-\infty, \infty)$. That is, the function $\exp(x)$ is its own derivative. We will have much more to say about this interesting property of the exponential function in Chapter 6.

Example From our work above we know that

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

for all x in $(-\infty, \infty)$. Now

$$\int_0^x \sin(t) dt = -\cos(t) \Big|_0^x = -\cos(x) + 1$$

for any x . However, we also know that

$$\begin{aligned} \int_0^x \sin(t) dt &= \int_0^x \left(\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} \right) dt \\ &= \sum_{n=0}^{\infty} \int_0^x \frac{(-1)^n t^{2n+1}}{(2n+1)!} dt \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+2}}{(2n+2)(2n+1)!} \Big|_0^x \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+2)!} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n}}{(2n)!}. \end{aligned}$$

Hence

$$\begin{aligned}\cos(x) &= 1 - \int_0^x \sin(t) dt \\ &= 1 - \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n}}{(2n)!} \\ &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}\end{aligned}$$

for all x in $(-\infty, \infty)$. Thus we have found a power series representation of $\cos(x)$ on $(-\infty, \infty)$. In particular, $\cos(x)$ is analytic on $(-\infty, \infty)$.

To close this section we note that a power series representation of a function about a specific point c is unique. To see this, suppose

$$f(x) = \sum_{n=0}^{\infty} a_n(x - c)^n \tag{5.7.15}$$

on $(c - R, c + R)$, where $R > 0$ is the radius of convergence of the power series. We need to show that the coefficients a_n , $n = 0, 1, 2, \dots$, are uniquely determined by f . To start,

$$\begin{aligned}f(c) &= \sum_{n=0}^{\infty} a_n(c - c)^n \\ &= a_0 + a_1(c - c) + a_2(c - c)^2 + a_3(c - c)^3 + \dots \\ &= a_0,\end{aligned} \tag{5.7.16}$$

so

$$a_0 = f(c). \tag{5.7.17}$$

Next,

$$f'(c) = \sum_{n=1}^{\infty} n a_n(c - c)^{n-1} = a_1, \tag{5.7.18}$$

so

$$a_1 = f'(c). \tag{5.7.19}$$

For a_2 we have

$$f''(c) = \sum_{n=2}^{\infty} n(n-1) a_n(c - c)^{n-2} = 2a_2, \tag{5.7.20}$$

so

$$a_2 = \frac{f''(c)}{2}. \tag{5.7.21}$$

In general, for $k = 0, 1, 2, \dots$,

$$f^{(k)}(c) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(c-c)^{n-k} = k!a_k, \quad (5.7.22)$$

from which it follows that

$$a_k = \frac{f^{(k)}(c)}{k!}. \quad (5.7.23)$$

As a consequence, the power series representation of f about c is uniquely determined by the values of the derivatives of f at c .

Proposition Suppose

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n \quad (5.7.24)$$

on $(c-R, c+R)$, where $R > 0$ is the radius of convergence of the power series. Then

$$a_n = \frac{f^{(n)}(c)}{n!} \quad (5.7.25)$$

for $n = 0, 1, 2, \dots$

Note that the coefficients a_n as given by (5.7.25) are the same as the coefficients used in the definition of the Taylor polynomial of f at c . This observation leads immediately to the question of extending Taylor polynomials to Taylor series, the topic of Section 5.8.

Our final example of this section illustrates how (5.7.25) may be used to find the derivatives of f at c if we already know a power series representation for f about c .

Example In a previous example we saw that if

$$f(x) = \frac{1}{1-x},$$

then

$$f(x) = \sum_{n=0}^{\infty} x^n$$

for all x in $(-1, 1)$. In this series, the coefficient of x^n is 1 for all n , so, by the previous proposition,

$$1 = \frac{f^{(n)}(0)}{n!}$$

for $n = 0, 1, 2, \dots$. That is, $f^{(n)}(0) = n!$ for all n .

Problems

1. For each of the following power series, find the interval of convergence and determine the behavior of the series at the endpoints of the interval. State clearly where the series converges absolutely, where it converges conditionally, and where it diverges. Also, for each series write out the first 5 nonzero terms.

(a)
$$\sum_{n=1}^{\infty} \frac{x^n}{n}$$

(b)
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$$

(c)
$$\sum_{n=1}^{\infty} n x^n$$

(d)
$$\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

(e)
$$\sum_{n=0}^{\infty} \frac{(x-2)^{n+1}}{n+1}$$

(f)
$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{3^n}$$

(g)
$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^2}$$

(h)
$$\sum_{n=1}^{\infty} n^3 x^{2n}$$

2. For each of the following power series, find the interval of convergence and determine the behavior of the series at the endpoints of the interval. State clearly where the series converges absolutely, where it converges conditionally, and where it diverges. Also, for each series write out the first 5 nonzero terms.

(a)
$$\sum_{n=0}^{\infty} \frac{(x-3)^n}{n!}$$

(b)
$$\sum_{n=1}^{\infty} \frac{x^n}{n 5^n}$$

(c)
$$\sum_{n=0}^{\infty} \frac{x^{2n}}{2^n}$$

(d)
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} (x-6)^n}{n 3^n}$$

(e)
$$\sum_{n=1}^{\infty} \frac{x^{2n+1}}{n}$$

(f)
$$\sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{2n}}{(2n)!}$$

(g)
$$\sum_{n=1}^{\infty} 3^n x^n$$

(h)
$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}$$

3. (a) Using the fact that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

for $-1 < x < 1$, find a power series representation about 0 for

$$f(x) = \frac{1}{1+x}$$

on $(-1, 1)$.

- (b) Use your result from (a) to find $f^{(35)}(0)$.

- (c) Use your result from (a) to find a power series representation about 0 for

$$\int_0^x \frac{1}{1+t} dt$$

on $(-1, 1)$. Determine where the series converges absolutely, where it converges conditionally, and where it diverges.

- (d) Use your result from (c) to find an infinite series representation for

$$\int_0^1 \frac{1}{1+t} dt.$$

Use this series to estimate the integral with an error of no more than 0.001 in absolute value.

4. Use the power series representations of $\sin(x)$ and $\cos(x)$ about 0 to prove the following identities.

(a) $\sin(-x) = -\sin(x)$

(b) $\cos(-x) = \cos(x)$

(c) $\frac{d}{dx} \sin(x) = \cos(x)$

(d) $\frac{d}{dx} \cos(x) = -\sin(x)$

5. Using the power series representation of $\cos(x)$ about 0, find an infinite series representation of $\cos(1)$. Use the infinite series to estimate $\cos(1)$ with an error of no more than 0.000001.

6. Use the fact that

$$\frac{d}{dx} \left(\frac{1}{1-x} \right) = \frac{1}{(1-x)^2}$$

to find a power series representation about 0 for

$$g(x) = \frac{1}{(1-x)^2}.$$

Find the interval of convergence for this power series and determine the behavior of the series at the endpoints.

7. Use your result from Problem 6 to evaluate

$$\sum_{n=1}^{\infty} \frac{n}{2^{n-1}}.$$

8. (a) A fair coin is tossed repeatedly. In Section 1.3 we saw that the probability that a head appears for the first time on the n th toss is

$$P_n = \frac{1}{2^n}$$

for $n = 1, 2, 3, \dots$. The average number of tosses before the first head appears is then given by

$$A = \sum_{n=1}^{\infty} nP_n.$$

Use Problem 7 to find A .

- (b) A manufacturer of circuit boards tests every board as it comes off the assembly line. If the probability that a board passes the test is p and the probability that it fails is $q = 1 - p$, then the probability that n boards are tested before the first defective board is encountered is $P_n = p^{n-1}q$. The average number of boards tested before finding a defective one is then

$$A = \sum_{n=1}^{\infty} nP_n.$$

Find A .