**Difference Equations**
**to**
**Differential Equations**

In the last section we saw that we could demonstrate the convergence of a series $\sum_{n=1}^{\infty} a_n$, where $a_n \geq 0$ for all $n$, by showing that $a_n$ approaches 0 as $n \to \infty$ as fast as the terms of another series with nonnegative terms which is already known to converge. Both of the techniques developed in Section 5.4, the comparison test and the limit comparison test, proved to be very useful; however, they both suffer from the drawback of requiring that we first find a series of known behavior which allows for the proper comparison with the series under consideration. In this section we shall consider another test for convergence, the *ratio test*, which determines whether or not the terms of a series are approaching 0 at a rate sufficient for the series to converge without reference to any other series. Although this test does not require knowledge of any other series, it has the limitation of being inconclusive in certain circumstances. Unfortunately, there is no single test for convergence which is useful under all conditions.

The ratio test determines if the terms of a given series are approaching 0 at a rate sufficient for convergence by considering the ratio between successive terms of the series. Specifically, suppose $a_n > 0$ for $n = 1, 2, 3, \ldots$ and

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \rho. \tag{5.5.1}$$

If $\rho < 1$, then there is an integer $N$ and a number $r$ with $\rho < r < 1$ such that

$$\frac{a_{n+1}}{a_n} < r \tag{5.5.2}$$

for all $n > N$. Then

$$a_{n+1} < r a_n \tag{5.5.3}$$

for all $n > N$, so

$$a_{N+2} < r a_{N+1},$$
$$a_{N+3} < r a_{N+2} < r^2 a_{N+1},$$
$$a_{N+4} < r a_{N+3} < r^3 a_{N+1},$$

and, in general, for any integer $m > 2$,

$$a_{N+m} < r^{m-1} a_{N+1}. \tag{5.5.4}$$

That is,

$$\left| \frac{a_{N+m}}{r^{m-1}} \right| < a_{N+1} \tag{5.5.5}$$

for $m = 2, 3, 4, \ldots$. Letting $n = N + m$, in which case $m = n - N$, we have

$$\left| \frac{a_n}{r^{n-N-1}} \right| < a_{N+1} \tag{5.5.6}$$

for all $n > N + 1$. Thus $a_n$ is $O(r^{n-N-1})$. Moreover,

$$\sum_{n=1}^{\infty} r^{n-N-1} = r^{-N} \sum_{n=1}^{\infty} r^{n-1} \tag{5.5.7}$$

converges since $\sum_{n=1}^{\infty} r^{n-1}$ is a geometric series and $0 < r < 1$. Thus $\sum_{n=1}^{\infty} a_n$ converges by the limit comparison test.

Now suppose $\rho > 1$, in which we include the possibility that $\rho = \infty$. Then there is an integer $N$ such that

$$\frac{a_{n+1}}{a_n} > 1 \tag{5.5.8}$$

for all $n > N$. Hence $a_{n+1} > a_n$ for all $n > N$, and so

$$a_{N+1} < a_{N+2} < a_{N+3} < a_{N+4} < \cdots . \tag{5.5.9}$$

In particular, $a_n > a_{N+1}$ for $n = N + 2, N + 3, N + 4, \ldots$. It follows that either $\lim_{n \to \infty} a_n$ does not exist or

$$\lim_{n \to \infty} a_n > a_{N+1} > 0. \tag{5.5.10}$$

Thus $\sum_{n=1}^{\infty} a_n$ diverges by the $n$th term test for divergence.

We now summarize the above results.

**Ratio Test**     Suppose $a_n > 0$ for $n = 1, 2, 3, \ldots$ and

$$\rho = \lim_{n \to \infty} \frac{a_{n+1}}{a_n}. \tag{5.5.11}$$

Then $\sum_{n=1}^{\infty} a_n$ converges if $\rho < 1$ and diverges if $\rho > 1$.

The examples below will show that the ratio test is inconclusive if $\rho = 1$. Namely, the third example considers a divergent series for which $\rho = 1$ and the fourth example considers a convergent series for which $\rho = 1$. Hence some other test will be necessary to determine the behavior of any series for which the ratio test yields $\rho = 1$.

**Example**     For the series

$$\sum_{n=1}^{\infty} \frac{n}{3^n},$$

if we let

$$a_n = \frac{n}{3^n},$$

$n = 1, 2, 3, \ldots$, then

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{n+1}{3^{n+1}}}{\frac{n}{3^n}} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)\left(\frac{3^n}{3^{n+1}}\right) = \frac{1}{3}\lim_{n\to\infty}\left(1 + \frac{1}{n}\right) = \frac{1}{3}.$$

Thus $\rho < 1$ and the series converges.

**Example**    For the series

$$\sum_{n=1}^{\infty} \frac{5^n}{n+2},$$

if we let

$$a_n = \frac{5^n}{n+2},$$

$n = 1, 2, 3, \ldots$, then

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{5^{n+1}}{n+3}}{\frac{5^n}{n+2}} = \lim_{n\to\infty} \left(\frac{n+2}{n+3}\right)\left(\frac{5^{n+1}}{5^n}\right) = 5\lim_{n\to\infty} \frac{1 + \frac{2}{n}}{1 + \frac{3}{n}} = 5.$$

Thus $\rho > 1$ and the series diverges.

**Example**    For the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n},$$

the ratio test yields

$$\rho = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1 + \frac{1}{n}} = 1.$$

This shows that it is possible for a series to diverge when $\rho = 1$.

**Example**    For the convergent $p$-series

$$\sum_{n=1}^{\infty} \frac{1}{n^2},$$

the ratio test yields

$$\rho = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 = \lim_{n\to\infty} \left(\frac{1}{1 + \frac{1}{n}}\right)^2 = 1.$$

This shows that is possible for a series to converge when $\rho = 1$.

**Example**   For the series

$$\sum_{n=1}^{\infty} \frac{3^n}{n!},$$

if we let

$$a_n = \frac{3^n}{n!},$$

$n = 1, 2, 3, \ldots$, then

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\dfrac{3^{n+1}}{(n+1)!}}{\dfrac{3^n}{n!}} = \lim_{n\to\infty} \left(\frac{n!}{(n+1)!}\right)\left(\frac{3^{n+1}}{3^n}\right) = \lim_{n\to\infty} \frac{3}{n+1} = 0.$$

Thus $\rho < 1$ and the series converges.

**Problems**

1. For each of the following infinite series, decide whether the series converges or diverges and explain your answer.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$ 

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{n!}$ 

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{7^{n+2}}{(n+1)!}$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n!}$ 

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{4}{n2^n}$

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$ 

(h) $\displaystyle\sum_{n=1}^{\infty} \frac{\pi^n}{5n+2}$

2. For each of the following infinite series, decide whether the series converges or diverges and explain your answer.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{4n-2}}$ 

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{3}{5^n}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n+5}}{(2n)!}$ 

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{n!n!}{(2n)!}$ 

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{3n+2}{7^{2n}}$

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{3n+1}{2n-1}$ 

(h) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{5^n} - \frac{3^n}{5^n}\right)$

3. Define

$$f(x) = \sum_{n=1}^{\infty} \frac{x^{2n}}{n}.$$

(a) Find the domain of $f$. That is, find all values of $x$ for which the series

$$\sum_{n=1}^{\infty} \frac{x^{2n}}{n}$$

converges.

(b) Plot an approximation to the graph of $f$ on the domain found in (a) using

$$f(x) \approx \sum_{n=1}^{100} \frac{x^{2n}}{n}.$$

4. Define

$$g(t) = \sum_{n=0}^{\infty} \frac{t^{2n}}{n!}.$$

(a) Find the domain of $g$. That is, find all values of $t$ for which the series

$$\sum_{n=0}^{\infty} \frac{t^{2n}}{n!}$$

converges.

(b) Plot an approximation to the graph of $g$ on the interval $[-2, 2]$ using

$$g(t) \approx \sum_{n=0}^{50} \frac{t^{2n}}{n!}.$$

5. Suppose the terms of the series $\sum_{n=1}^{\infty} a_n$ satisfy the difference equation

$$a_{n+1} = \frac{(n+1)a_n}{2n}$$

with $a_1 = 10$. Does this series converge or diverge? Explain.

6. Suppose, for $n = 1, 2, 3, \ldots$, $a_n \geq 0$ and

$$\alpha = \lim_{n \to \infty} \sqrt[n]{a_n}.$$

Show that $\sum_{n=1}^{\infty} a_n$ converges if $\alpha < 1$ and diverges if $\alpha > 1$. This result is known as the *root test*.