Difference Equations
to
Differential Equations

Section 5.4

Infinite Series: The Comparison Test

In this section we continue our discussion of the convergence properties of infinite series. Now that we have two classes of series, namely, the geometric series and the p-series, for which classification as either convergent or divergent is relatively easy, it is reasonable to develop tests for convergence based on comparing a given series with a series of known behavior. We will see this idea first in the *comparison test*, which we will later generalize with the *limit comparison test*.

To begin, suppose $\sum_{n=1}^{\infty} a_n$ is a convergent series with $a_n \geq 0$ for all n and $\sum_{n=1}^{\infty} b_n$ is a series with $0 \leq b_n \leq a_n$ for all n. Let

$$L = \sum_{n=1}^{\infty} a_n.$$

If s_n is the nth partial sum of $\sum_{n=1}^{\infty} a_n$ and t_n is the nth partial sum of $\sum_{n=1}^{\infty} b_n$, then

$$t_n \leq s_n \tag{5.4.1}$$

for $n = 1, 2, 3, \ldots$. Since $a_n \geq 0$ for all n, the sequence $\{s_n\}$ is increasing; hence

$$s_n \leq L \tag{5.4.2}$$

for all n. Since $b_n \geq 0$, $n = 1, 2, 3, \ldots$, $\{t_n\}$ is also an increasing sequence which, by (5.4.1) and (5.4.2), is bounded above by L. Hence $\lim\limits_{n\to\infty} t_n$ exists, showing that $\sum_{n=1}^{\infty} b_n$ converges. Moreover, from (5.4.1),

$$\sum_{n=1}^{\infty} b_n \leq \sum_{n=1}^{\infty} a_n. \tag{5.4.3}$$

Now suppose that $\sum_{n=1}^{\infty} a_n$ is a divergent series with $a_n \geq 0$ for all n and $\sum_{n=1}^{\infty} b_n$ is a series with $a_n \leq b_n$ for all n. If s_n is the nth partial sum of $\sum_{n=1}^{\infty} a_n$ and t_n is the nth partial sum of $\sum_{n=1}^{\infty} b_n$, then

$$t_n \geq s_n \tag{5.4.4}$$

for $n = 1, 2, 3, \ldots$. Since $a_n \geq 0$ for all n, $\{s_n\}$ is an increasing sequence; thus, since $\sum_{n=1}^{\infty} a_n$ diverges, it follows that

$$\lim_{n\to\infty} s_n = \infty.$$

Hence, from (5.4.4), and the fact that $\{t_n\}$ is also an increasing sequence, we have

$$\lim_{n\to\infty} t_n = \infty,$$

that is, $\sum_{n=1}^{\infty} b_n$ diverges.

The preceding results are summarized in the comparison test.

Comparison Test Suppose $a_n \geq 0$ for $n = 1, 2, 3, \ldots$ and $\sum_{n=1}^{\infty} a_n$ converges. If $0 \leq b_n \leq a_n$ for $n = 1, 2, 3, \ldots$, then $\sum_{n=1}^{\infty} b_n$ converges and

$$\sum_{n=1}^{\infty} b_n \leq \sum_{n=1}^{\infty} a_n. \tag{5.4.5}$$

Suppose $a_n \geq 0$ for $n = 1, 2, 3, \ldots$ and $\sum_{n=1}^{\infty} a_n$ diverges. If $a_n \leq b_n$ for $n = 1, 2, 3, \ldots$, then $\sum_{n=1}^{\infty} b_n$ diverges.

In other words, if the terms of the series $\sum_{n=1}^{\infty} b_n$ are nonnegative and smaller than the terms of a series which converges, then $\sum_{n=1}^{\infty} b_n$ must converge; if the terms of the series $\sum_{n=1}^{\infty} b_n$ are larger than those of a divergent series with nonnegative terms, then $\sum_{n=1}^{\infty} b_n$ must diverge.

Example The series

$$\sum_{m=1}^{\infty} \frac{1}{n^2+1}$$

converges since

$$0 < \frac{1}{n^2+1} < \frac{1}{n^2}$$

for $n = 1, 2, 3, \ldots$, and

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

is a convergent series (namely, a p-series with $p = 2$).

Example The series

$$\sum_{n=1}^{\infty} \frac{1}{2n-1}$$

diverges since

$$\frac{1}{2n-1} > \frac{1}{2n} > 0$$

for $n = 1, 2, 3, \ldots$, and

$$\sum_{n=1}^{\infty} \frac{1}{2n}$$

is a divergent series (since it is a multiple of the harmonic series).

Example The series

$$\sum_{n=1}^{\infty} \frac{\sin^2(n)}{n^2}$$

converges since

$$0 \leq \frac{\sin^2(n)}{n^2} \leq \frac{1}{n^2}$$

for $n = 1, 2, 3, \ldots$, and

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

is, as in a previous example, a convergent series.

Example The series

$$\sum_{n=1}^{\infty} \frac{n+1}{n^2}$$

diverges since

$$\frac{n+1}{n^2} = \left(\frac{n+1}{n}\right)\frac{1}{n} > \frac{1}{n} > 0$$

for $n = 1, 2, 3, \ldots$, and

$$\sum_{n=1}^{\infty} \frac{1}{n},$$

the harmonic series, is a divergent series.

Example The series

$$\sum_{n=1}^{\infty} \frac{1}{n3^n}$$

converges since

$$0 < \frac{1}{n3^n} \leq \frac{1}{3^n}$$

for $n = 1, 2, 3, \ldots$, and

$$\sum_{n=1}^{\infty} \frac{1}{3^n}$$

is a convergent series (namely, a geometric series with ratio $\frac{1}{3}$).

We know that if a series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim\limits_{n\to\infty} a_n = 0$; however, we have seen numerous examples, such as the harmonic series, which show the latter condition is not sufficient to guarantee convergence. To ensure that a series with nth term $a_n \geq 0$ satisfying $\lim\limits_{n\to\infty} a_n = 0$ converges, we need additional information about the rate at which the terms are approaching 0; namely, we need to know that a_n approaches 0 fast enough to guarantee that the sequence of partial sums, although increasing, is nevertheless bounded.

The problem lies in determining how to measure rates of convergence to 0 and how to decide what rates are sufficient for convergence. We have already seen that the comparison test, by comparing a_n with the terms of a series of known behavior, provides one way to measure whether or not a_n is approaching 0 fast enough for the sum to converge. However, finding a series to use for the comparison can be somewhat tricky, even when the behavior of the series is relatively obvious. For example, the series

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}$$

should converge, since for large values of n there is very little difference between

$$\frac{1}{n^2 - 1}$$

and

$$\frac{1}{n^2}.$$

But a direct comparison will not work, since

$$\frac{1}{n^2 - 1} > \frac{1}{n^2}$$

for $n = 2, 3, 4, \ldots$. Hence it would be useful to have a method for describing the rate at which a sequence $\{a_n\}$ converges to 0 and a test which exploits this description. We will supply the former with the next definition, a version of the "O" and "o" notation adapted for sequences, and the latter with the limit comparison test.

Definition If $\{a_n\}$ and $\{b_n\}$ are sequences with

$$\lim_{n \to \infty} \frac{b_n}{a_n} = 0, \tag{5.4.6}$$

then we say b_n is $o(a_n)$. If $\{a_n\}$ and $\{b_n\}$ are sequences and there exists an integer N and a constant M such that

$$\left| \frac{b_n}{a_n} \right| \leq M \tag{5.4.7}$$

for all $n > N$, then we say b_n is $O(a_n)$.

Similar to our earlier results, if

$$\lim_{n \to \infty} \frac{b_n}{a_n}$$

exists, then b_n is $O(a_n)$ (see Problem 5). Analogous to our earlier use of this notation, if

$$\lim_{n \to \infty} a_n = 0,$$

$$\lim_{n\to\infty} b_n = 0,$$

and b_n is $o(a_n)$, then b_n is approaching 0 faster than a_n as $n \to \infty$; if

$$\lim_{n\to\infty} a_n = 0,$$

$$\lim_{n\to\infty} b_n = 0,$$

and b_n is $O(a_n)$, then b_n is approaching 0 at least as fast as a_n as $n \to \infty$. Of course, if b_n is $o(a_n)$, then b_n is also $O(a_n)$.

Example $\frac{1}{3n^2+4}$ is $O\left(\frac{1}{n^2}\right)$ since

$$\lim_{n\to\infty} \frac{\dfrac{1}{3n^2+4}}{\dfrac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{3n^2+4} = \lim_{n\to\infty} \frac{1}{3+\dfrac{4}{n^2}} = \frac{1}{3}.$$

Example $\frac{1}{n^4+6}$ is $o\left(\frac{1}{n^3}\right)$ since

$$\lim_{n\to\infty} \frac{\dfrac{1}{n^4+6}}{\dfrac{1}{n^3}} = \lim_{n\to\infty} \frac{n^3}{n^4+6} = \lim_{n\to\infty} \frac{\dfrac{1}{n}}{1+\dfrac{6}{n^4}} = 0.$$

Now consider two series, $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, where $a_n > 0$ and $b_n \geq 0$ for all n, b_n is $O(a_n)$, and $\sum_{n=1}^{\infty} a_n$ converges. Then there is an integer N and a constant M such that

$$\frac{b_n}{a_n} \leq M \tag{5.4.8}$$

for all $n > N$. Hence

$$b_n \leq M a_n \tag{5.4.9}$$

for all $n > N$, so the series

$$\sum_{n=N+1}^{\infty} b_n$$

converges by comparison with the convergent series

$$\sum_{n=N+1}^{\infty} M a_n.$$

Thus $\sum_{n=1}^{\infty} b_n$ is also a convergent series.

Next consider two series, $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, where $a_n \geq 0$ and $b_n > 0$ for all n, a_n is $O(b_n)$, and $\sum_{n=1}^{\infty} a_n$ diverges. Then there exists an integer N and a constant $M > 0$ such that

$$\frac{a_n}{b_n} \leq M \tag{5.4.10}$$

for all $n > N$. Hence

$$b_n \geq \frac{a_n}{M} \tag{5.4.11}$$

for all $n > N$, so the series

$$\sum_{n=N+1}^{\infty} b_n$$

diverges by comparison with the divergent series

$$\sum_{n=N+1}^{\infty} \frac{a_n}{M}.$$

Thus $\sum_{n=1}^{\infty} b_n$ is also a divergent series.

The preceding results are summarized in the limit comparison test.

Limit Comparison Test Suppose $a_n > 0$ for $n = 1, 2, 3, \ldots$ and $\sum_{n=1}^{\infty} a_n$ converges. If $b_n \geq 0$ for $n = 1, 2, 3, \ldots$ and b_n is $O(a_n)$, then $\sum_{n=1}^{\infty} b_n$ also converges. Suppose $a_n \geq 0$ for $n = 1, 2, 3, \ldots$ and $\sum_{n=1}^{\infty} a_n$ diverges. If $b_n > 0$ for $n = 1, 2, 3, \ldots$ and a_n is $O(b_n)$, then $\sum_{n=1}^{\infty} b_n$ also diverges.

In other words, if the terms of the series $\sum_{n=1}^{\infty} b_n$ are nonnegative and approach 0 at least as fast as the terms of a convergent series with positive terms, then $\sum_{n=1}^{\infty} b_n$ converges; if the terms of a divergent series are nonnegative and approach 0 at least as fast as the terms of the series $\sum_{n=1}^{\infty} b_n$, which are positive, then $\sum_{n=1}^{\infty} b_n$ diverges.

Example Consider the series

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}.$$

As mentioned above, we would expect this series to behave very much like the convergent series

$$\sum_{n=2}^{\infty} \frac{1}{n^2}.$$

Now

$$\lim_{n \to \infty} \frac{\dfrac{1}{n^2 - 1}}{\dfrac{1}{n^2}} = \lim_{n \to \infty} \frac{n^2}{n^2 - 1} = \lim_{n \to \infty} \frac{1}{1 - \dfrac{1}{n^2}} = 1,$$

so $\frac{1}{n^2-1}$ is $O\left(\frac{1}{n^2}\right)$. Hence

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}$$

converges by the limit comparison test.

Example Consider the series

$$\sum_{n=1}^{\infty} \frac{2n^2}{n^3+2}.$$

Since the nth term of this series is a rational function of n with a numerator of degree 2 and a denominator of degree 3, we might expect this series to behave much like the divergent series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

Now

$$\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{2n^2}{n^3+2}} = \lim_{n\to\infty} \frac{n^3+2}{2n^3} = \lim_{n\to\infty} \left(\frac{1}{2} + \frac{1}{n^3}\right) = \frac{1}{2},$$

so $\frac{1}{n}$ is $O\left(\frac{2n^2}{n^3+2}\right)$. Hence

$$\sum_{n=1}^{\infty} \frac{2n^2}{n^3+2}.$$

diverges by the limit comparison test.

Problems

1. For each of the following infinite series, decide whether the series converges or diverges and explain your answer.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{3}{2^n}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3+2}$

   (c) $\displaystyle\sum_{n=3}^{\infty} \frac{2}{\sqrt{n-2}}$ (d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$

   (e) $\displaystyle\sum_{n=1}^{\infty} \pi^{-n}$ (f) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2(n)}{n^4}$

   (g) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{3^n} - n^{-\frac{3}{2}}\right)$ (h) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+2}$

2. For each of the following infinite series, decide whether the series converges or diverges and explain your answer.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sin^2(n)}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+2n}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{3n-1}{2n^4+2}$ (d) $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n2^n}$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)3^{-n}}{n}$

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{2n}$

(g) $\displaystyle\sum_{n=23}^{\infty} \frac{3n^5+1}{13n^7-2}$

(h) $\displaystyle\sum_{n=2}^{\infty} \frac{n+1}{n^2-2}$

3. (a) Give an example of a convergent series $\sum_{n=1}^{\infty} a_n$ and a divergent series $\sum_{n=1}^{\infty} b_n$ with the property that $b_n \le a_n$ for all n.
   (b) Give an example of a divergent series $\sum_{n=1}^{\infty} a_n$ and a convergent series $\sum_{n=1}^{\infty} b_n$ with the property that $a_n \le b_n$ for all n.
   (c) Comment on why the comparison test does not apply to the series in (a) and (b).
4. Explain why
$$\int_1^{\infty} \frac{1}{4x^5-2}\,dx$$
converges.
5. Show that if
$$\lim_{n\to\infty} \frac{b_n}{a_n}$$
exists, then b_n is $O(a_n)$.