Difference Equations to Differential Equations

Section 5.2

Taylor's Theorem

The goal of this section is to prove that if P_n is the nth order Taylor polynomial for a function f at a point c, then, under suitable conditions, the remainder function

$$R_n(h) = f(c+h) - P_n(c+h) \tag{5.2.1}$$

is $O(h^{n+1})$. This result is a consequence of *Taylor's theorem*, which we now state and prove.

Taylor's Theorem Suppose f is continuous on the closed interval $[a, b]$ and has $n+1$ continuous derivatives on the open interval (a, b). If x and c are points in (a, b), then

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + r_n(x), \tag{5.2.2}$$

where

$$r_n(x) = \int_c^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt. \tag{5.2.3}$$

That is, if P_n is the nth order Taylor polynomial for f at some point c in (a, b) and x is any point in (a, b), then

$$f(x) = P_n(x) + r_n(x), \tag{5.2.4}$$

where r_n is given by (5.2.3).

We will show that Taylor's theorem follows from the Fundamental Theorem of Integral Calculus combined with repeated applications of integration by parts. Let f be a function satisfying the conditions of the theorem. Since f is an antiderivative of f', by the Fundamental Theorem of Integral Calculus we have

$$f(x) - f(c) = \int_c^x f'(t)dt. \tag{5.2.5}$$

Hence

$$f(x) = f(c) + \int_c^x f'(t)dt, \tag{5.2.6}$$

which is the statement of Taylor's theorem when $n = 0$. For $n = 1$, we perform an integration by parts on the integral in (5.2.6) using

$$\begin{aligned} u &= f'(t) & dv &= dt \\ du &= f''(t) & v &= -(x-t). \end{aligned}$$

Note that this is not the most obvious choice for v (certainly, $v = t$ would be a simpler choice), but it is a valid choice and one that leads to the result we desire. Namely, this gives us

$$\begin{aligned} f(x) &= f(c) - f'(t)(x-t)\Big|_c^x + \int_c^x (x-t)f''(t)dt \\ &= f(c) + f'(c)(x-c) + \int_c^x (x-t)f''(t)dt, \end{aligned}$$

which is the statement of Taylor's theorem for the case $n = 1$. For $n = 2$, we perform another integration by parts using

$$\begin{aligned} u &= f''(t) \qquad dv = (x-t)dt \\ du &= f'''(t) \qquad v = -\frac{(x-t)^2}{2}, \end{aligned}$$

from which we obtain

$$\begin{aligned} f(x) &= f(c) + f'(c)(x-c) - \frac{f''(t)(x-t)^2}{2}\Big|_c^x + \int_c^x \frac{(x-t)^2}{2} f'''(t)dt \\ &= f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \int_c^x \frac{(x-t)^2}{2} f'''(t)dt. \end{aligned}$$

Similarly, we obtain Taylor's theorem for $n = 3$ by another integration by parts. This time we have

$$\begin{aligned} u &= f'''(t) \qquad dv = \frac{(x-t)^2}{2}\, dt \\ du &= f^{(4)}(t) \qquad v = -\frac{(x-t)^3}{3!}, \end{aligned}$$

which yields

$$\begin{aligned} f(x) &= f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 - \frac{f'''(t)(x-t)^3}{3!}\Big|_c^x + \int_c^x \frac{(x-t)^3}{3!} f^{(4)}(t)dt \\ &= f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \int_c^x \frac{(x-t)^3}{3!} f^{(4)}(t)dt. \end{aligned}$$

From this we can see that, for any nonnegative integer n, performing integration by parts n times will yield

$$\begin{aligned} f(x) = f(c) &+ f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots \\ &+ \frac{f^{(n)}(c)}{n!}(x-c)^n + \int_c^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt, \end{aligned} \tag{5.2.7}$$

which is the general statement of Taylor's theorem.

In applying Taylor's theorem, it is seldom the case that the remainder term $r_n(x)$ can be evaluated exactly. In most cases, we try to find an upper bound for $|r_n(x)|$ so that we know what is the worst possible error that we could commit in approximating $f(x)$ by $P_n(x)$. For these purposes, there is an alternative formulation of the remainder term which is often more useful than the one given in Taylor's theorem.

Lagrange's form of the remainder term Using the same notation as in the statement of Taylor's theorem, there exists a number k between c and x such that

$$r_n(x) = \frac{f^{(n+1)}(k)(x-c)^{n+1}}{(n+1)!}. \tag{5.2.8}$$

To show this, we will assume $x > c$, the argument in the case $x < c$ being similar. So let u be the point where $f^{(n+1)}$ attains its maximum value on $[c, x]$ and let v be the point where $f^{(n+1)}$ attains its minimum value on $[c, x]$. Note that we know such points exist because we have assumed $f^{(n+1)}$ to be a continuous function on (a, b), and hence on $[c, x]$. Then we have

$$\frac{(x-t)^n}{n!} f^{(n+1)}(v) \le \frac{(x-t)^n}{n!} f^{(n+1)}(t) \le \frac{(x-t)^n}{n!} f^{(n+1)}(u) \tag{5.2.9}$$

for all t in $[c, x]$. Integrating each of the terms in (5.2.9) from c to x, we have

$$f^{(n+1)}(v) \int_c^x \frac{(x-t)^n}{n!}\, dt \le \int_c^x \frac{(x-t)^n}{n!} f^{(n+1)}(t) dt \le f^{(n+1)}(u) \int_c^x \frac{(x-t)^n}{n!}\, dt. \tag{5.2.10}$$

Now

$$\int_c^x \frac{(x-t)^n}{n!}\, dt = -\frac{(x-t)^{n+1}}{(n+1)!}\bigg|_c^x = \frac{(x-c)^{n+1}}{(n+1)!} \tag{5.2.11}$$

and

$$\int_c^x \frac{(x-t)^n}{n!} f^{(n+1)}(t) dt = r_n(x), \tag{5.2.12}$$

so (5.2.10) implies that

$$\frac{f^{(n+1)}(v)(x-c)^{n+1}}{(n+1)!} \le r_n(x) \le \frac{f^{(n+1)}(u)(x-c)^{n+1}}{(n+1)!}. \tag{5.2.13}$$

Finally, since

$$g(t) = \frac{f^{(n+1)}(t)(x-c)^{n+1}}{(n+1)!} \tag{5.2.14}$$

is a continuous function of t on the interval $[c, x]$, it follows from (5.2.13) and the Intermediate Value Theorem that there exists a number k in $[c, x]$ such that $g(k) = r_n(x)$, that is,

$$r_n(x) = \frac{f^{(n+1)}(k)(x-c)^{n+1}}{(n+1)!}, \tag{5.2.15}$$

which is (5.2.8).

Of course, we cannot calculate $r_n(x)$ exactly without knowing the value of k. However, if we can find a number M such that

$$|f^{(n+1)}(t)| \leq M \tag{5.2.16}$$

for all t between c and x, then (5.2.8) implies that

$$|r_n(x)| \leq \frac{M}{(n+1)!}|x-c|^{n+1}. \tag{5.2.17}$$

Hence, although usually we cannot hope to know the exact amount of error in our approximation, in this case we can at least find an upper bound for the size of the error.

We can now show that $r_n(x)$ is $O((x-c)^{n+1})$, or, equivalently, that

$$R_n(h) = f(c+h) - P_n(c+h) \tag{5.2.18}$$

is $O(h^{n+1})$. First choose $\epsilon > 0$ so that the interval $I = [c-\epsilon, c+\epsilon]$ is contained in the interval (a,b), and let M be the maximum value of $|f^{(n+1)}|$ on I. Then, using (5.2.17), we have

$$|r_n(x)| \leq \frac{M}{(n+1)!}|x-c|^{n+1} \tag{5.2.19}$$

for all x in I. Thus if $|h| \leq \epsilon$,

$$|R_n(h)| \leq \frac{M}{(n+1)!}|h|^{n+1}, \tag{5.2.20}$$

from which it follows that

$$\left|\frac{R_n(h)}{h^{n+1}}\right| \leq \frac{M}{(n+1)!}. \tag{5.2.21}$$

That is, $R_n(h)$ is $O(h^{n+1})$.

Proposition If f satisfies the conditions of Taylor's theorem and P_n is the nth order Taylor polynomial for f at c, then

$$R_n(h) = f(c+h) - P_n(h) \tag{5.2.22}$$

is $O(h^{n+1})$.

Of course, from our previous work we know that this statement implies that $R_n(h)$ is also $o(h^n)$.

With this proposition we may write

$$f(c+h) = f(c) + f'(c)h + \frac{f''(c)}{2!}h^2 + \cdots + \frac{f^{(n)}(c)}{n!}h^n + O(h^{n+1}), \tag{5.2.23}$$

or, in terms of $x = c + h$,

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + O((x-c)^{n+1}), \tag{5.2.24}$$

as long as f is $n+1$ times continuously differentiable on an open interval containing c.

Example Recall that the fifth order Taylor polynomial for $\sin(x)$ at 0 is

$$P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}.$$

Since in this case $P_5 = P_6$, we now know that

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + O(x^7).$$

More explicitly, since

$$\frac{d^7}{dx^7}\sin(x) = -\cos(x),$$

we have

$$\left|\frac{d^7}{dx^7}\sin(x)\right| \leq 1$$

for any value of x. Hence if

$$r_5(x) = \sin(x) - P_5(x),$$

then by (5.2.17) it follows that

$$|r_5(x)| \leq \frac{|x|^7}{7!}$$

for any value of x. For example,

$$|\sin(1) - P_5(1)| \leq \frac{1}{7!} = \frac{1}{5040} = 0.000198,$$

to 6 decimal places. That is, the error in approximating $\sin(1)$ by $P_5(1)$ is no more than 0.000198. In this case the error bound is very close to the actual error, for, to six decimal place accuracy,

$$\sin(1) = 0.841471$$

and

$$P_5(1) = 1 - \frac{1}{6} + \frac{1}{120} = 0.841667,$$

which gives an error of

$$\sin(1) - P_5(1)| = 0.000196.$$

Note that, in general, for any nonnegative integer n,

$$\left|\frac{d^{2n+3}}{dx^{2n+3}}\sin(x)\right| = |\cos(x)|.$$

Thus

$$\left|\frac{d^{2n+3}}{dx^{2n+3}}\sin(x)\right| \leq 1$$

for any value of x. Hence, using the fact that in this case $P_{2n+1} = P_{2n+2}$,

$$|\sin(x) - P_{2n+1}(x)| \leq \frac{|x|^{2n+3}}{(2n+3)!} \tag{5.2.25}$$

for any value of x. With this inequality we can determine the order necessary for a Taylor polynomial to give some desired level of accuracy for a particular approximation. For example, if we wish to estimate $\sin(1.4)$ with an error of less than 0.001 using a Taylor polynomial about 0, then (5.2.25) says we need only find a nonnegative integer n such that

$$\frac{1.4^{2n+3}}{(2n+3)!} < 0.001,$$

in which case $P_{2n+1}(1.4)$ will provide the desired approximation. For $n = 0, 1, 2,$ and 3, we have the following table:

n	$\frac{1.4^{2n+3}}{(2n+3)!}$
0	0.4573333
1	0.0448187
2	0.0020915
3	0.0000569

Hence the smallest value of n that will work is $n = 3$; thus to attain the desired level of accuracy we would use

$$P_7(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}.$$

Checking this to 7 decimal places, we find that

$$P_7(1.4) = 0.9853938$$

and

$$\sin(1.4) = 0.9854497$$

an error of only 0.0000559.

Example Combining our work from Section 5.1 about the Taylor polynomial of order 4 for $f(x) = \sqrt{x}$ at 1 with our new results, we now have

$$\sqrt{x} = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 - \frac{5}{128}(x-1)^4 + O((x-1)^5).$$

If, for example, we wanted to bound the error involved in using $P_4(1.6)$ as an estimate for $\sqrt{1.6}$, we would first note that

$$f^{(5)}(x) = \frac{105}{32x^{\frac{9}{2}}},$$

which is a decreasing function and hence is maximized on the interval $[1, 1.6]$ at $x = 1$. Thus

$$|f^{(5)}(x)| \leq \frac{105}{32}$$

for all x in $[1, 1.6]$. From (5.2.16) it follows that

$$|\sqrt{1.6} - P_4(1.6)| \leq \frac{\frac{105}{32}}{5!}|1.6 - 1|^5 = 0.0021263,$$

to 7 decimal places. Checking this on a calculator to 7 decimal places, we have

$$\sqrt{1.6} = 1.2649111$$

and

$$P_4(1.6) = 1.2634375,$$

showing that the error in this case is 0.0014736.

If f is indefinitely differentiable on an interval about a point c and P_n is the Taylor polynomial of order n for f at c, then it is frequently the case that $P_1, P_2, P_3, \ldots$ is a sequence of increasingly accurate approximating polynomials for f on some interval I containing the point c. Of course, unless f is itself a polynomial, there is no polynomial P_n in this sequence such that $f(x) = P_n(x)$ for all x in I. Nevertheless, there are many functions for which

$$f(x) = \lim_{n\to\infty} P_n(x) \tag{5.2.26}$$

for all x in some interval I. Such functions are said to be *analytic*. Since a polynomial is just the sum of a finite number of monomials, $\lim\limits_{n\to\infty} P_n(x)$ may be regarded as an infinite sum of monomials, an infinite polynomial. That is, if f is an analytic function, then

$$\begin{aligned} f(x) &= \lim_{n\to\infty} P_n(x) \\ &= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n. \end{aligned} \tag{5.2.27}$$

for all x in some interval I containing c. An infinite series of this type is called a *power series*. Since power series have many of the nice properties of polynomials, such as being easy to integrate, a representation of a function f by a power series in this manner can be extremely useful. Although we considered infinite series in Section 1.3, we will need a far more thorough discussion of them before we will be able fully to understand a series like (5.2.27). We will do this in Sections 5.3 through 5.6.

Problems

1. For each of the following functions f, find the Taylor polynomial of order 5 at the point c, use it to approximate $f(a)$, and find an upper bound for the absolute value of the error in the approximation.

   (a) $f(x) = \sin(x)$, $c = 0$, $a = 0.8$
   (b) $f(x) = \cos(x)$, $c = 0$, $a = 1.2$
   (c) $f(x) = \sin(2x)$, $c = 0$, $a = -0.5$
   (d) $f(x) = \sqrt{x}$, $c = 1$, $a = 1.5$
   (e) $f(x) = \sqrt{x}$, $c = 9$, $a = 10$
   (f) $f(x) = x^{\frac{3}{2}}$, $c = 1$, $a = 1.4$
   (g) $f(x) = \dfrac{1}{x}$, $c = 1$, $a = 0.8$
   (h) $f(x) = \sin(x)$, $c = 0$, $a = -1.2$

2. Use a Taylor polynomial to approximate $\sin(0.6)$ with an error of less than 0.0001.

3. Use a Taylor polynomial to approximate $\sin(-1.3)$ with an error of less than 0.00001.

4. Use a Taylor polynomial to approximate $\cos(1.2)$ with an error of less than 0.0001.

5. Find the Taylor polynomial of smallest order that will approximate $\sin(x)$ with an error of less than 0.0005 for all x in $[-2, 2]$.

6. Suppose L is a function defined on $(0, \infty)$ with $L(1) = 0$ and
$$L'(x) = \frac{1}{x}.$$
   (a) Find P_{10}, the Taylor polynomial of order 10 for L at 1.
   (b) Use P_{10} to approximate $L(1.5)$. Find an upper bound for the absolute value of the error of this approximation.
   (c) Find the Taylor polynomial of smallest degree that will approximate $L(x)$ with an error less than 0.0005 for all x in $[1, 1.5]$.

7. Suppose E is a function defined on $(-\infty, \infty)$ with $E(0) = 1$ and $E'(x) = E(x)$ for all x.
   (a) Find P_{10}, the Taylor polynomial of order 10 for E at 0.
   (b) Use P_{10} to approximate $E(1)$.
   (c) Given that $|E(x)| < 3^x$ for all $x > 0$, find an upper bound for the absolute value of the error in the approximation in part (b).
   (d) Find the Taylor polynomial of smallest degree that will approximate $E(x)$ with an error less than 0.0001 for all x in $[0, 2]$.

8. Let P_9 be the 9th order Taylor polynomial for $f(x) = \sin(x)$ at 0. Use P_9 to approximate
$$\int_0^3 \frac{\sin(x)}{x}\, dx.$$

9. (a) Find the 6th order Taylor polynomial for $f(x) = \sin(x^2)$ at 0. How is it related to the 3rd order Taylor polynomial for $g(x) = \sin(x)$ at 0?

   (b) Find the 7th order Taylor polynomial at 0 for

$$h(x) = \int_0^x \sin(t^2)dt.$$

   How is it related to your answer in (a)?