Difference Equations
_to_
Differential Equations

Section 4.4

Using the Fundamental Theorem

As we saw in Section 4.3, using the Fundamental Theorem of Integral Calculus reduces the problem of evaluating a definite integral to the problem of finding an antiderivative. Unfortunately, finding antiderivatives, even for relatively simple functions, cannot be done as routinely as the computation of derivatives. For example, suppose we let $f(x) = \sin(x)$, $g(x) = x$, and

$$h(x) = \frac{f(x)}{g(x)} = \frac{\sin(x)}{x}.$$

Then, since we know the derivative of f and we know the derivative of g, it is a simple matter to find the derivative of h using the quotient rule. However, knowing the antiderivatives of f and g in no way helps us find the antiderivative of h. In fact, it has been shown that the antiderivative of h is not expressible in terms of any finite combination of algebraic and elementary transcendental functions. Because of results like this, many of the definite integrals that are encountered in applications cannot be evaluated using the Fundamental Theorem of Integral Calculus; instead, they must be approximated using numerical techniques such as those we studied in Section 4.2. Of course, when antiderivatives are available, the Fundamental Theorem is the best way to evaluate an integral. To this end, we will investigate, in this section and in the next, techniques for evaluating definite integrals by finding antiderivatives and applying the Fundamental Theorem.

Before we begin, we need to introduce some additional notation and terminology. First of all, we will call the collection of all antiderivatives of a given function f the *general antiderivative* of f. For example, if $f(x) = 3x^2$, then the general antiderivative of f is given by $F(x) = x^3 + c$, where c is an arbitrary constant.

Second, since the Fundamental Theorem of Calculus draws a close connection between antiderivatives and definite integrals, it is customary to borrow the notation for the general antiderivative from the notation for the definite integral. Hence the general antiderivative of a function f with respect to the variable x is denoted by

$$\int f(x)dx. \tag{4.4.1}$$

This is usually referred to as the *indefinite integral* of f with respect to x. Thus the terms indefinite integral and general antiderivative are synonymous, and from this point on we will prefer the former to the latter.

Example In this notation, we write

$$\int 3x^2 dx = x^3 + c,$$

where c is assumed to be an arbitrary constant.

Since finding an indefinite integral involves reversing the process of differentiation, we can rewrite our basic results about derivatives in terms of indefinite integrals. Hence we have the following list of integration formulas:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c \text{ (where } n \neq -1 \text{ is a rational number),} \tag{4.4.2}$$

$$\int \sin(x)dx = -\cos(x) + c, \tag{4.4.3}$$

$$\int \cos(x)dx = \sin(x) + c, \tag{4.4.4}$$

$$\int \sec^2(x)dx = \tan(x) + c, \tag{4.4.5}$$

$$\int \csc^2(x)dx = -\cot(x) + c, \tag{4.4.6}$$

$$\int \sec(x)\tan(x)dx = \sec(x) + c, \tag{4.4.7}$$

$$\int \csc(x)\cot(x)dx = -\csc(x) + c. \tag{4.4.8}$$

Note that each one of these formulas may be verified by checking that the derivative of the right-hand side is equal to the function inside the integral sign on the left-hand side. Also notice that we have not used any special techniques to find these results; rather, we know these formulas only because they are the inverses of differentiation formulas that we learned in Chapter 3. Thus, for example, we know that

$$\int \sec^2(x)dx = \tan(x) + c,$$

but we do not know, nor do we even know how to begin to find, $\int \sec(x)dx$, which would at first seem to be an easier problem.

The following proposition is a consequence of the corresponding basic properties of differentiation.

Proposition If the indefinite integrals of f and g exist, then

$$\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx \tag{4.4.9}$$

and

$$\int (f(x) - g(x))dx = \int f(x)dx - \int g(x)dx. \tag{4.4.10}$$

Moreover, for any constant k,

$$\int kf(x)dx = k\int f(x)dx. \tag{4.4.11}$$

Example Using (4.4.2) with the results of the previous proposition, we have

$$\int (5x^3 - 3x + 2)dx = 5\int x^3 dx - 3\int x dx + 2\int 1 dx = \frac{5}{4}x^4 - \frac{3}{2}x^2 + 2x + c.$$

It is worth noting that $\int 1dx$ is typically denoted simply by $\int dx$.

Example Using (4.4.2),

$$\int \frac{1}{\sqrt{t}}\, dt = \int t^{-\frac{1}{2}} dt = \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + c = 2\sqrt{t} + c.$$

Example Using (4.4.2), (4.4.4), and (4.4.9), we have

$$\begin{aligned}\int \left(\cos(x) + \frac{4}{x^2}\right) dx &= \int \cos(x)dx + 4\int x^{-2}dx \\ &= \sin(x) - 4x^{-1} + c \\ &= \sin(x) - \frac{4}{x} + c.\end{aligned}$$

Sometimes the indefinite integral of a function, although not itself in the list (4.4.2) through (4.4.8), may be found with the use of some intelligent guessing. For example, $F(x) = \sin(2x)$ is not an antiderivative of $f(x) = \cos(2x)$ since $F'(x) = 2\cos(2x)$. However, since F' and f differ only by a factor of 2, we can correct for this by dividing F by 2. That is,

$$\int \cos(2x)dx = \frac{1}{2}\sin(2x) + c.$$

Again, as with all indefinite integrals, you may verify this result by differentiation.

Example To find $\int 3\sin(4x)dx$, we might begin with a guess using $F(x) = -3\cos(4x)$. However, $F'(x) = 12\sin(3x)$, which differs from the function we are integrating by a factor of 4. Thus, dividing our initial guess by 4, we have

$$\int 3\sin(4x)dx = -\frac{3}{4}\cos(4x) + c.$$

Example To find $\int \sqrt{2t+3}\, dt$, we might begin with a guess using

$$F(t) = \frac{(2t+3)^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{3}(2t+3)^{\frac{3}{2}}.$$

However,

$$F'(t) = \sqrt{2t+3}\frac{d}{dt}(2t+3) = 2\sqrt{2t+3},$$

so we need to divide our guess by 2. Hence

$$\int \sqrt{2t+3}\ dt = \frac{1}{3}(2t+3)^{\frac{3}{2}} + c.$$

Example To find

$$\int \frac{1}{\sqrt{3z+1}}\ dz,$$

we might start with an initial guess of

$$F(z) = \frac{(3z+1)^{\frac{1}{2}}}{\frac{1}{2}} = 2\sqrt{3z+1}.$$

Since

$$F'(z) = \frac{3}{\sqrt{3z+1}},$$

we find that

$$\int \frac{1}{\sqrt{3z+1}}\ dz = \frac{2}{3}\sqrt{3z+1} + c.$$

Thus, for example,

$$\int_0^5 \frac{1}{\sqrt{3z+1}}\ dz = \frac{2}{3}\sqrt{3z+1}\ \Big|_0^5 = \frac{8}{3} - \frac{2}{3} = 2.$$

The common thread in the previous examples is the need to modify an initial guess because of the chain rule. For example, $F(x) = \sin(2x)$ is not an antiderivative of $f(x) = \cos(2x)$ because the chain rule comes into play when differentiating F, resulting in an extra factor of 2. This process of reversing the chain rule can be taken a step further to help evaluate integrals in even more complicated situations. For example, consider the indefinite integral

$$\int 2x\sqrt{1+x^2}\ dx.$$

The key to evaluating this integral is recognizing that the factor $2x$ is the derivative of the function inside the square root. That is, if we let

$$f(u) = \sqrt{u}$$

and

$$g(x) = 1 + x^2,$$

then

$$\int 2x\sqrt{1+x^2}\ dx = \int f(g(x))g'(x)dx.$$

Thus we are trying to find an antiderivative of a function which is in the form of the result from a chain rule differentiation. Now if F is an antiderivative of f, then, using the chain rule,

$$\frac{d}{dx}F(g(x)) = F'(g(x))g'(x) = f(g(x))g'(x).$$

Hence, thinking of u as $1 + x^2$, we really only need to find the antiderivative of f with respect to u. Now

$$\int f(u)du = \int \sqrt{u}\ du = \frac{2}{3}u^{\frac{3}{2}} + c,$$

so, substituting $1 + x^2$ back in for u, we should have

$$\int 2x\sqrt{1+x^2}\ dx = \frac{2}{3}(1+x^2)^{\frac{3}{2}} + c.$$

You should check this result by differentiation, noting in particular that the factor of $2x$ comes from the use of the chain rule.

In general, if F is an antiderivative of f and $u = g(x)$ is some differentiable function of x, then, by the chain rule,

$$\frac{d}{dx}F(u) = F'(u)\frac{du}{dx} = f(u)\frac{du}{dx}. \tag{4.4.12}$$

Writing this as an integration formula, we have

$$\int f(u)\frac{du}{dx}\ dx = F(u) + c = \int f(u)du. \tag{4.4.13}$$

This technique to help find indefinite integrals is called *integration by substitution.*

Example To find

$$\int 2x\sin(x^2)dx,$$

we should let $u = x^2$. Then

$$\frac{du}{dx} = 2x,$$

so, using (4.4.13) with $f(u) = \sin(u)$,

$$\begin{aligned}\int 2x\sin(x^2)dx &= \int \sin(u)\frac{du}{dx}\ dx \\ &= \int \sin(u)du \\ &= -\cos(u) + c \\ &= -\cos(x^2) + c.\end{aligned}$$

We summarize this technique as follows.

Integration by substitution To evaluate an indefinite integral of the form

$$\int f(g(x))g'(x)dx, \tag{4.4.14}$$

we may make the substitution $u = g(x)$. We then have

$$\int f(g(x))g'(x)dx = \int f(u)\frac{du}{dx}\,dx = \int f(u)du. \tag{4.4.15}$$

Of course, this technique will work only if we know an antiderivative for f. Indeed, all we have done is replace one indefinite integral with another, with the hope that the new integral will be simpler than the original. In our notation, we can think of the transition from

$$\int f(g(x))g'(x)dx$$

to

$$\int f(u)du$$

as replacing $g(x)$ by u and $g'(x)dx$ by du. Thus in practice we often denote the process of substitution by writing

$$\begin{aligned} u &= g(x) \\ du &= g'(x)dx \end{aligned} \tag{4.4.16}$$

and directly substituting into the integral

$$\int f(g(x))g'(x)dx$$

to obtain the integral

$$\int f(u)du.$$

We will illustrate this in the next examples.

Example To evaluate the indefinite integral

$$\int \frac{2x}{\sqrt{2+x^2}}\,dx,$$

we may let $u = 2 + x^2$. Then

$$\frac{du}{dx} = 2x,$$

which we write in the form

$$du = 2xdx.$$

Substituting, we have

$$\int \frac{2x}{\sqrt{2+x^2}}\,dx = \int \frac{1}{\sqrt{u}}\,du = 2\sqrt{u} + c = 2\sqrt{2+x^2} + c. \tag{4.4.17}$$

Example From (4.4.17), it is easy to see, after dividing through by 2, that

$$\int \frac{x}{\sqrt{2+x^2}}\,dx = \sqrt{2+x^2} + c.$$

We could also see this directly when making the substitution. Namely, if we let $u = 2+x^2$, then $du = 2xdx$ may be written as

$$\frac{1}{2}\,du = xdx.$$

Hence, if we substitute $2+x^2$ for u and $\frac{1}{2}du$ for xdx, we obtain

$$\int \frac{2x}{\sqrt{2+x^2}}\,dx = \int \frac{\frac{1}{2}}{\sqrt{u}}\,du = \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = \frac{1}{2}(2\sqrt{u}) + c = \sqrt{2+x^2} + c.$$

Example To evaluate the indefinite integral

$$\int 5x\cos(x^2)dx,$$

we may make the substitution

$$\begin{aligned} u &= x^2 \\ du &= 2xdx. \end{aligned}$$

Then

$$\frac{1}{2}\,du = xdx,$$

so we have

$$\int 5x\cos(x^2)dx = \frac{5}{2}\int \cos(u)du = \frac{5}{2}\sin(u) + c = \frac{5}{2}\sin(x^2) + c.$$

Example To evaluate the indefinite integral

$$\int \tan(3x)\sec^2(3x)dx,$$

we may make the substitution

$$\begin{aligned} u &= \tan(3x) \\ du &= 3\sec^2(3x)dx. \end{aligned}$$

Then

$$\frac{1}{3}\,du = \sec^2(3x)dx,$$

so we have

$$\int \tan(3x)\sec^2(3x)dx = \frac{1}{3}\int udu = \frac{1}{6}u^2 + c = \frac{1}{6}\tan^2(3x) + c.$$

Now suppose we want to evaluate the definite integral

$$\int_a^b f(g(x))g'(x)dx.$$

If F is an antiderivative of f, then we know that

$$\int f(g(x))g'(x)dx = F(g(x)) + c. \tag{4.4.18}$$

Hence

$$\int_a^b f(g(x))g'(x)dx = F(g(x))\Big|_a^b = F(g(b)) - F(g(a)). \tag{4.4.19}$$

Now we also have

$$\int_{g(a)}^{g(b)} f(u)du = F(u)\Big|_{g(a)}^{g(b)} = F(g(b)) - F(g(a)). \tag{4.4.20}$$

Putting (4.4.19) and (4.4.20) together, we see that

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du. \tag{4.4.21}$$

That is, similar to our work with indefinite integrals, we may evaluate the definite integral

$$\int_a^b f(g(x))g'(x)dx$$

by making a substitution

$$\begin{aligned} u &= g(x) \\ du &= g'(x)dx, \end{aligned} \tag{4.4.22}$$

the only difference being that in the definite integral we must also change the limits of integration. Note that the new limits of integration correspond to the range of values for u given that x is ranging from a to b.

Example To evaluate

$$\int_0^1 \frac{x^2}{(1+x^3)^2}\, dx,$$

we may make the substitution

$$\begin{aligned} u &= 1 + x^3 \\ du &= 3x^2. \end{aligned}$$

Then

$$\frac{1}{3}\, du = x^2 dx$$

and u varies from

$$1 + 0^3 = 1$$

to

$$1 + 1^3 = 2$$

as x varies from 0 to 1, so

$$\int_0^1 \frac{x^2}{(1+x^3)^2}\, dx = \frac{1}{3}\int_1^2 \frac{1}{u^2}\, du = -\frac{1}{3}\frac{1}{u}\bigg|_1^2 = -\frac{1}{6} + \frac{1}{3} = \frac{1}{6}.$$

Example To evaluate

$$\int_0^{\frac{\pi}{2}} \sin^2(x)\cos(x)dx,$$

we may make the substitution

$$\begin{aligned} u &= \sin(x) \\ du &= \cos(x)dx. \end{aligned}$$

Then u varies from 0 to 1 as x varies from 0 to $\frac{\pi}{2}$, so

$$\int_0^{\frac{\pi}{2}} \sin^2(x)\cos(x)dx = \int_0^1 u^2 du = \frac{1}{2}u^3\bigg|_0^1 = \frac{1}{3}.$$

Example To evaluate

$$\int_0^{\frac{\pi}{2}} \cos^3(x)\sin(x)dx,$$

we may make the substitution

$$\begin{aligned} u &= \cos(x) \\ du &= -\sin(x)dx. \end{aligned}$$

Then $-du = -\sin(x)dx$ and u varies from 1 to 0 as x varies from 0 to $\frac{\pi}{2}$, so

$$\int_0^{\frac{\pi}{2}} \cos^3(x)\sin(x)dx = -\int_1^0 u^3 du = \int_0^1 u^3 du = \frac{1}{4}u^4\bigg|_0^1 = \frac{1}{4}.$$

Example So far all of our examples of substitution have involved reversing the results of the chain rule. However, substitutions can be useful in other situations as well. For example, to evaluate

$$\int_0^3 x\sqrt{1+x}\,dx,$$

the substitution

$$\begin{aligned} u &= 1+x \\ du &= dx \end{aligned}$$

turns out to be useful for rearranging the integral into a form which can be evaluated. Namely, since $u = 1+x$ implies that $x = u-1$, we may substitute to obtain

$$\begin{aligned} \int_0^3 x\sqrt{1+x}\,dx &= \int_1^4 (u-1)\sqrt{u}\,du \\ &= \int_1^4 (u^{\frac{3}{2}} - u^{\frac{1}{2}})du \\ &= \frac{2}{5}u^{\frac{5}{2}}\Big|_1^4 - \frac{2}{3}u^{\frac{3}{2}}\Big|_1^4 \\ &= \left(\frac{64}{5} - \frac{2}{5}\right) - \left(\frac{16}{3} - \frac{2}{3}\right) \\ &= \frac{116}{15}. \end{aligned}$$

We will continue the discussion of techniques for using the Fundamental Theorem of Integral Calculus in Section 4.5.

Problems

1. Evaluate the following indefinite integrals.

   (a) $\int (x^3 + 3x - 6)dx$ (b) $\int (3t^2 - 4t + 5)dt$

   (c) $\int \frac{1}{x^4}\,dx$ (d) $\int \left(3z - \frac{4}{z^2}\right)dz$

   (e) $\int \frac{12}{\sqrt{t}}\,dt$ (f) $\int \frac{3}{\sqrt{2+x}}\,dx$

   (g) $\int 7\sqrt{x+5}\,dx$ (h) $\int (\sin(\theta) - 2\cos(\theta))d\theta$

2. Evaluate the following indefinite integrals.

   (a) $\int \sin(3x)dx$ (b) $\int \cos(4x)dx$

   (c) $\int \sqrt{3t-1}\,dt$ (d) $\int \frac{4}{\sqrt{1+5z}}\,dz$

(e) $\int 7\sec^2(2x)dx$ (f) $\int 3\sec(4x)\tan(4x)dx$

(g) $\int 2\csc^2(7x)dx$ (h) $\int \sin(4x+1)dx$

3. Evaluate the following indefinite integrals.

(a) $\int 6x\sqrt{1+3x^2}\ dx$ (b) $\int 4x^3\cos(x^4)dx$

(c) $\int x^2(3+x^3)^{10}dx$ (d) $\int \frac{7x}{\sqrt{4+3x^2}}\ dx$

(e) $\int 4t\sin(t^2)dt$ (f) $\int 7z\cos(3z^2+1)dz$

(g) $\int \sin^3(t)\cos(t)dt$ (h) $\int 4\cos^4(3t)\sin(3t)dt$

4. Evaluate the following indefinite integrals.

(a) $\int \frac{\sin(\sqrt{x})}{\sqrt{x}}\ dx$ (b) $\int \sec^2(x)\tan^2(x)dx$

(c) $\int \sec^3(4x)\tan(4x)dx$ (d) $\int \sin(\theta)\cos(\theta)d\theta$

(e) $\int \frac{\sin(x)}{\cos^2(x)}\ dx$ (f) $\int \frac{\cos(3t)}{\sqrt{1+\sin(3t)}}\ dt$

(g) $\int t\sqrt{t-2}\ dt$ (h) $\int \frac{z}{\sqrt{z+1}}\ dz$

5. Evaluate the following definite integrals.

(a) $\int_0^1 (4x^2-3x-5)dx$ (b) $\int_0^1 \frac{1}{\sqrt{3x+1}}\ dx$

(c) $\int_0^{\frac{\pi}{4}} 3\sin(2x)dx$ (d) $\int_1^5 \sqrt{2t-1}\ dt$

(e) $\int_{-\frac{\pi}{12}}^0 6\sec(3t)\tan(3t)dt$ (f) $\int_0^2 \frac{1}{(7z+6)^2}\ dz$

(g) $\int_0^2 x\sqrt{x^2+1}\ dx$ (h) $\int_0^{\pi} \sin^4(t)\cos(t)dt$

6. Evaluate the following definite integrals.

(a) $\int_{-1}^1 \frac{5x^2}{(x^3+2)^2}\ dx$ (b) $\int_0^2 \frac{3x}{\sqrt{x^2+1}}\ dx$

(c) $\int_0^{\sqrt{\pi}} 3x\sin(x^2)dx$ (d) $\int_{-\frac{\pi}{2}}^0 \cos^2(t)\sin(t)dt$

(e) $\int_0^{\frac{\pi}{2}} \sin^3(2t)\cos(2t)dt$ (f) $\int_0^1 5x(2+x^2)^{10}dx$

(g) $\displaystyle\int_{-1}^{1} x(1+x^2)^{25}dx$

(h) $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^2(u)\tan(u)du$

7. Evaluate the following definite integrals.

(a) $\displaystyle\int_0^3 \frac{x}{\sqrt{x+1}}\,dx$

(b) $\displaystyle\int_0^1 x^3\sqrt{x^2+1}\,dx$

(c) $\displaystyle\int_{-\frac{\pi}{12}}^{\frac{\pi}{24}} \tan^4(4x)\sec^2(4x)dx$

(d) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cot(t)\csc^2(t)dt$

(e) $\displaystyle\int_0^1 4x(1+x)^{25}dx$

(f) $\displaystyle\int_0^{\frac{\pi}{3}} \frac{\sin(2\theta)}{\cos^3(2\theta)}\,d\theta$

(g) $\displaystyle\int_1^5 5u\sqrt{2u-1}\,du$

(h) $\displaystyle\int_0^{\pi} \sin^5(w)\cos(w)dw$

8. Find the area beneath one arch of the curve $y = 4\sin(6t)$.

9. (a) Plot the graph of $y = \sin^2(x)\cos(x)$ over the interval $[0, \pi]$.

   (b) Find the area of the region beneath the graph of $y = \sin^2(x)\cos(x)$ over the interval $\left[0, \frac{\pi}{2}\right]$.

   (c) Verify that

$$\int_0^{\pi} \sin^2(x)\cos(x)dx = 0$$

   and justify your result geometrically.