Difference Equations
to
Differential Equations

Section 4.1

The Definite Integral

As we discussed in Section 1.1, and mentioned again at the beginning of Section 3.1, there are two basic problems in calculus. In Chapter 3 we considered one of these, the problem of finding tangent lines to curves in the plane; we are now ready to turn to the second, quadrature, the problem of finding the area of a region in the plane. Although at first these problems would seem to have no connection, in Section 4.3 we shall see that Fundamental Theorem of Calculus relates them in an interesting and useful way. This theorem, first fully utilized by Newton and Leibniz, reveals that the problem of quadrature involves reversing the process of differentiation; as a consequence, the facility we developed in Chapter 3 for handling derivatives will be very helpful in many basic quadrature problems.

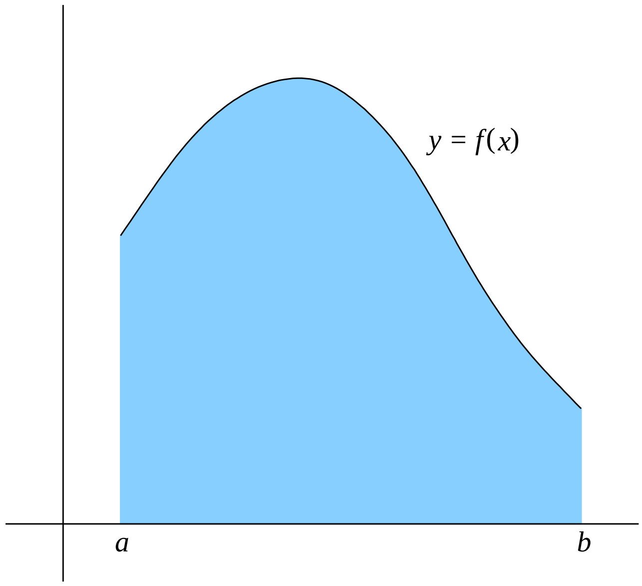

Figure 4.1.1 Region beneath the graph of $y = f(x)$ and over the interval $[a, b]$

As illustrated in Figure 4.1.1, our basic example for studying quadrature will be the problem of finding the area of a region R in the plane which is bounded above by the graph of a continuous function f and below by an interval $[a, b]$ on the x-axis. Later we will see how to extend our techniques to more complicated planar regions. Recall that in Section 1.1 we considered the problem of finding the area of the unit circle. In that case, we attacked the problem by approximating the area of the circle by the area of inscribed regular polygons, which were themselves divided into triangles. We used these to find the area of the circle by taking the limit of the areas of the inscribed polygons as the number

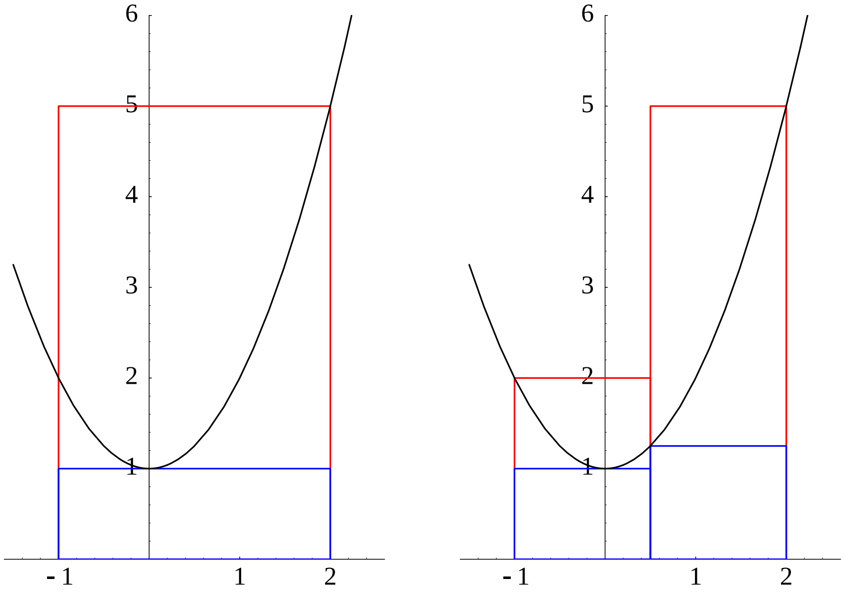

Figure 4.1.2 Inscribed and circumscribed rectangles for $f(x) = x^2 + 1$

of sides went to infinity. Here we will see that it is sufficient to use rectangles, rather than triangles, as our units of approximation. That is, we will approximate the area of the desired region by the area of rectangles and then ask about the limit as the number of rectangles used in the approximation goes to infinity. We begin with an example.

Example Consider the region R beneath the graph of the function $f(x) = x^2 + 1$ and above the interval $[-1, 2]$ on the x-axis. Let A be the area of R. If R_1 is the rectangle with base on the interval $[-1, 2]$ and height $f(2) = 5$, then, since 5 is the maximum value of f on $[-1, 2]$, R_1 contains R. We call R_1 a *circumscribed rectangle* for the region R. Hence the area of R is less than the area of R_1, showing that $A \leq 15$. Similarly, if R_2 is the rectangle with base on the interval $[-1, 2]$ and height $f(0) = 1$, then, since 1 is the minimum value of f on $[-1, 2]$, R contains R_2. We call R_2 an *inscribed rectangle* for the region R. Hence the area of R is greater than the area of R_2, showing that $A \geq 3$. See the figure on the left in Figure 4.1.2.

At this point we know that

$$3 \leq A \leq 15.$$

To improve our approximations for A, we begin by subdividing the interval $[-1, 2]$ into two equal intervals, namely $[-1, 0.5]$ and $[0.5, 2]$. If A_1 is the area of the region beneath the curve over the interval $[-1, 0.5]$, we can construct inscribed and circumscribed rectangles as we did in the last paragraph and obtain bounds for the area of A_1. Indeed, the rectangle with base on $[-1, 0.5]$ and height $f(-1) = 2$ circumscribes this region, while the rectangle with base on $[-1, 0.5]$ and height $f(0) = 1$ is inscribed in it. Hence we have

$$\frac{3}{2} \leq A_1 \leq 3.$$

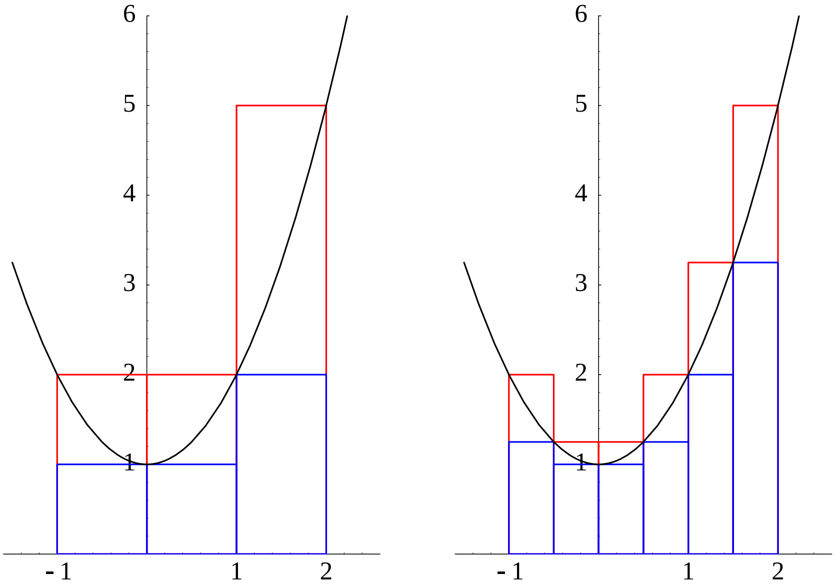

Figure 4.1.3 Inscribed and circumscribed rectangles for $f(x) = x^2 + 1$

Moreover, the region beneath the curve over the interval $[0.5, 2]$ is circumscribed by a rectangle of height $f(2) = 5$ and has inscribed within it a rectangle of height $f(0.5) = 1.25$. So if A_2 is the area of this region, we have

$$\frac{15}{8} \leq A_2 \leq \frac{15}{2}.$$

Since

$$A = A_1 + A_2,$$

putting these last two results together gives us

$$\frac{27}{8} \leq A \leq \frac{21}{2},$$

an improvement on our previous approximation. See the figure on the right in Figure 4.1.2.

To improve our approximation further, divide $[-1, 2]$ into three equal intervals: $[-1, 0]$, $[0, 1]$, and $[1, 2]$. You should check that the heights of the inscribed rectangles over these intervals are 1, 1, and 2, respectively. Since each rectangle has a base of length 1, we have

$$A \geq (1)(1) + (1)(1) + (2)(1) = 4.$$

Moreover, the heights of the circumscribed rectangles are 2, 2, and 5, respectively, and so

$$A \leq (2)(1) + (2)(1) + (5)(1) = 9.$$

Hence we now have

$$4 \leq A \leq 9.$$

See the figure on the left in Figure 4.1.3.

It is clear that we can approximate A using inscribed and circumscribed rectangles for any number of intervals. For example, you might check that if we use six intervals of equal length we would have

$$A \geq \left(\frac{5}{4}\right)\left(\frac{1}{2}\right) + (1)\left(\frac{1}{2}\right) + (1)\left(\frac{1}{2}\right) + \left(\frac{5}{4}\right)\left(\frac{1}{2}\right) + (2)\left(\frac{1}{2}\right) + \left(\frac{13}{4}\right)\left(\frac{1}{2}\right) = \frac{39}{8}$$

and

$$A \leq (2)\left(\frac{1}{2}\right) + \left(\frac{5}{4}\right)\left(\frac{1}{2}\right) + \left(\frac{5}{4}\right)\left(\frac{1}{2}\right) + (2)\left(\frac{1}{2}\right) + \left(\frac{13}{4}\right)\left(\frac{1}{2}\right) + (5)\left(\frac{1}{2}\right) = \frac{59}{8}$$

showing that

$$4.875 \leq A \leq 7.375,$$

(see the figure on the right in Figure 4.1.2). Continuing in this manner, subdividing the interval $[-1, 2]$ into smaller and smaller intervals, we would expect that we could approximate A to any desired level of accuracy. Moreover, we would expect that the area of the inscribed rectangles would increase toward A as the number of intervals increases, and that the area of the circumscribed rectangles would decrease toward A. Put another way, we might think of the area A as the unique number which is at once larger than the area of any set of inscribed rectangles and smaller than the area of any set of circumscribed rectangles. We will use this idea as the basis for our definition of the *definite integral*.

The definite integral

We now want to take the ideas of the previous example and develop a general procedure which, when applied to the appropriate function, will yield the area of certain types of regions in the plane. To do so, we require some preliminary terminology and notation.

Let f be a function defined on an interval $[a, b]$. We will not require that f be positive on $[a, b]$, although it will be necessary to require $f(x) \geq 0$ for all x in in order to talk about the area between the graph of f and the interval $[a, b]$, as in the previous example. However, we will assume that f is *bounded* on $[a, b]$; that is, we assume there exist numbers m and M such that $m \leq f(x) \leq M$ for all x in $[a, b]$. In particular, by the Extreme Value Theorem, f is bounded if f is continuous on $[a, b]$. We will return to the problem of unbounded functions in Section 4.7.

We call a set $P = \{x_0, x_1, \ldots, x_n\}$ a *partition* of the interval $[a, b]$ if

$$a = x_0 < x_1 < x_2 < \ldots < x_n = b.$$

Such a partition P divides $[a, b]$ into n intervals, $[x_{i-1}, x_i]$, of lengths

$$\Delta x_i = x_i - x_{i-1},$$

where $i = 1, 2, 3, \ldots, n$. For each such interval $[x_{i-1}, x_i]$, let M_i be the smallest number such that $f(x) \leq M_i$ for all x in $[x_{i-1}, x_i]$ and let m_i be the largest number such that $f(x) \geq m_i$ for all x in $[x_{i-1}, x_i]$. Note that if f is continuous on $[a, b]$, then M_i is the maximum value of f on $[x_{i-1}, x_i]$ and m_i is the minimum value of f on $[x_{i-1}, x_i]$, both of which are guaranteed to exist by the Extreme Value Theorem. If f is not continuous, properties of bounded sets of real numbers, alluded to in our discussion of bounded sequences in Section 1.2, nevertheless guarantee the existence of the values M_i and m_i. Also, note that if $f(x) \geq 0$ for all x in $[x_{i-1}, x_i]$, then, in the language of our previous example, the rectangle with base $[x_{i-1}, x_i]$ and height M_i is a circumscribed rectangle and the rectangle with base $[x_{i-1}, x_i]$ and height m_i is an inscribed rectangle.

Now let

$$U(f, P) = M_1 \Delta x_1 + M_2 \Delta x_2 + \cdots + M_n \Delta x_n = \sum_{i=1}^{n} M_i \Delta x_i, \tag{4.1.1}$$

the *upper sum* of f with respect to the partition P, and

$$L(f, P) = m_1 \Delta x_1 + m_2 \Delta x_2 + \cdots + m_n \Delta x_n = \sum_{i=1}^{n} m_i \Delta x_i, \tag{4.1.2}$$

the *lower sum* of f with respect to the partition P. Note that we always have

$$L(f, P) \leq U(f, P). \tag{4.1.3}$$

Also, if $f(x) \geq 0$ for all x in $[a, b]$, then $U(f, P)$ is the sum of the areas of the circumscribed rectangles for the partition P and $L(f, P)$ is the sum of the areas of the inscribed rectangles. In that case, if A is the area beneath the graph of f and above the interval , we would expect that we could make $U(f, P)$ and $L(f, P)$ arbitrarily close to A. This would imply that A is the only number with the property that

$$L(f, P) \leq A \leq U(f, P) \tag{4.1.4}$$

for all partitions P. This is the motivation for the following definition.

Definition Using the above notation, we say a function f is *integrable* on an interval $[a, b]$ if there exists a unique number I such that

$$L(f, P) \leq I \leq U(f, P) \tag{4.1.5}$$

for all partitions P of $[a, b]$. If f is integrable on $[a, b]$, we call I the *definite integral* of f on $[a, b]$, which we denote

$$I = \int_a^b f(x)dx. \tag{4.1.6}$$

Example Consider again our example of finding the area of the region beneath the graph of $f(x) = x^2 + 1$ and above the interval $[-1, 2]$ on the x-axis. Let P_n denote the partition using $n + 1$ equally spaced points (giving us n intervals of equal length). For examples,

$$P_2 = \{-1, 0.5, 2\}$$

and

$$P_6 = \{-1, -0.5, 0, 0.5, 1, 1.5, 2\}.$$

Our work above shows that, in our current notation,

$$U(f, P_6) = 7.375$$

and

$$L(f, P_6) = 4.875.$$

Using 100 intervals, and a computer to ease the computations, we find that

$$U(f, P_{100}) = 6.075$$

and

$$L(f, P_{100}) = 5.925,$$

where the results have been rounded to three decimal places. This shows that if f is integrable on $[-1, 2]$, then

$$5.925 \leq \int_{-1}^{2} (x^2 + 1)dx \leq 6.075.$$

Of course, we expect f to be integrable, and for the value of the definite integral to be the sought for area under the graph.

It is not easy to verify directly from the definition that a given function is integrable on some interval. However, it may be shown that any continuous function is integrable. The reasons for this are rather technical, but we can give some feeling for why this should be so. Suppose f is continuous on $[a, b]$ and let $P_n = \{x_0, x_1, x_2, \ldots, x_n\}$ denote the partition of $[a, b]$ using $n + 1$ equally spaced points. Let M_i and m_i be as defined above, and let

$$\Delta x = \Delta x_i = \frac{b - a}{n}$$

be the length of the intervals $[x_{i-1}, x_i]$, $i = 1, 2, 3, \ldots, n$. Given any number $\epsilon > 0$, we can choose n large enough (equivalently, Δx small enough) so that $M_i - m_i < \epsilon$ for $i = 1, 2, 3, \ldots, n$. This fact is a consequence of the continuity of f on $[a, b]$, although it requires a deeper property of continuous functions on closed intervals known as *uniform continuity.* We then have

$$\begin{aligned} 0 &\leq U(f, P_n) - L(f, P_n) \\ &= \sum_{i=1}^{n} M_i \Delta x - \sum_{i=1}^{n} m_i \Delta x \\ &= \sum_{i=1}^{n} (M_i - m_i)\Delta x < \sum_{i=1}^{n} \epsilon \Delta x \\ &= n\epsilon\Delta x = \epsilon(b - a). \end{aligned} \tag{4.1.7}$$

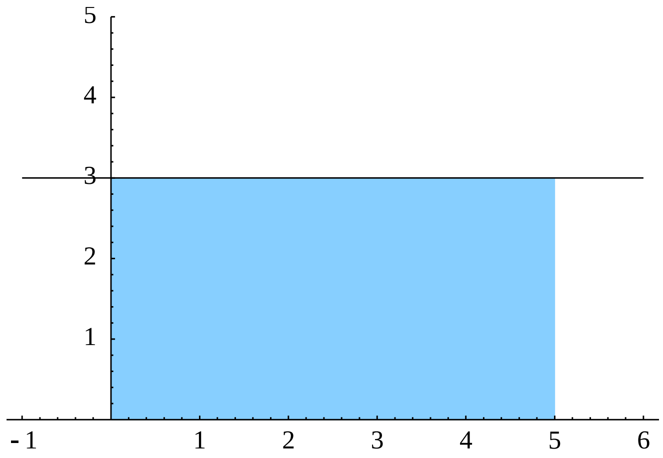

Figure 4.1.4 Region beneath $y = 3$ over the interval $[0, 5]$

Since ϵ may be made arbitrarily small, it follows that the difference between upper sums and lowers sums may be made arbitrarily small, and hence that there must be only one number which is between the upper and lower sums for all possible partitions.

Proposition If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.

Example We now know that $f(x) = x^2 + 1$ is integrable on $[-1, 2]$.

Although our motivation for this section has been the computation of area, we have not actually defined the term. We do so now for the special case we have been considering.

Definition Given an integrable function f with $f(x) \geq 0$ for all x in an interval $[a, b]$, let R be the region in the plane bounded above by the curve $y = f(x)$, below by the interval $[a, b]$ on the x-axis, and on the sides by the vertical lines $x = a$ and $x = b$. Then we define the *area* A of R to be

$$A = \int_a^b f(x)dx. \tag{4.1.8}$$

Example Of course, we should verify that the above definition of area agrees with our previous notion of area. For example, if $f(x) = 3$ for all x in $[0, 5]$, then the region beneath the graph of f and above the x-axis is a rectangle with base of length 5 and height of 3, as shown in Figure 4.1.4. Hence we should have

$$\int_0^5 3dx = 15.$$

To verify this, let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be any partition of $[0, 5]$. Then on any interval $[x_{i-1}, x_i]$, $i = 1, 2, 3, \ldots, n$, the maximum value of f is $M_i = 3$ and the minimum value of f is $m_i = 3$. Hence

$$U(f, P) = L(f, P) = \sum_{i=1}^{n} 3\Delta x_i = 3\sum_{i=1}^{n} \Delta x_i = (3)(5) = 15,$$

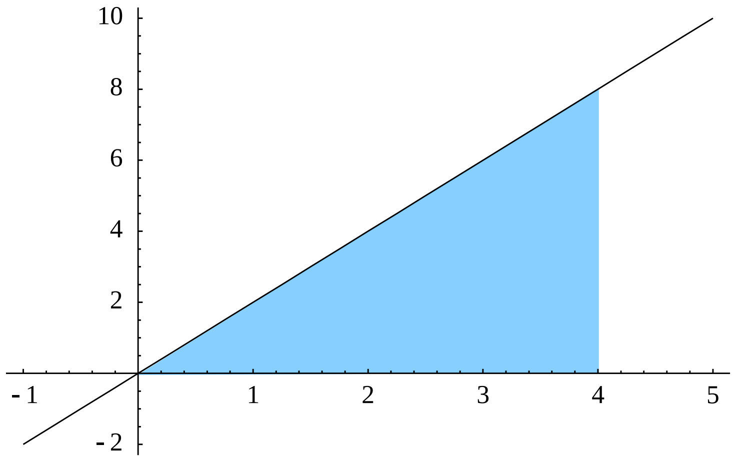

Figure 4.1.5 Region beneath $y = 2x$ over the interval $[0, 4]$

where we have used the fact that the sum of the lengths of the partition intervals must equal the length of the entire interval . Thus $I = 15$ is the only number satisfying

$$L(f, P) \leq I \leq U(f, P)$$

for all partitions P, and so

$$\int_0^5 3dx = 15,$$

as expected.

Note that the previous example could be generalized to show that for any constant c and any interval $[a, b]$,

$$\int_a^b cdx = c(b - a). \tag{4.1.9}$$

Example To verify another previously known area, consider the function $f(x) = 2x$ on the interval $[0, 4]$. Then the region beneath the graph of f and above the interval $[0, 4]$ on the x-axis is a triangle with base of length 4 and height 8, as shown in Figure 4.1.5. Thus it has area

$$\frac{1}{2}(4)(8) = 16,$$

and so we should have

$$\int_0^4 2xdx = 16.$$

To verify this, let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be a partition of $[0, 4]$ and let m_i and M_i be the minimum and maximum values, respectively, of f on $[x_{i-1}, x_i]$, $i = 1, 2, 3, \ldots, n$. Since f is an increasing function on $[0, 4]$, we have $m_i = f(x_{i-1})$ and $M_i = f(x_i)$. Thus

$$L(f, P) = \sum_{i=1}^{n} f(x_{i-1})\Delta x_i$$

and

$$U(f,P) = \sum_{i=1}^{n} f(x_i)\Delta x_i.$$

We will now use a technique which will be useful in the proof of the Fundamental Theorem of Calculus in Section 4.3. Let $F(x) = x^2$. Then $F'(x) = 2x$, so F is an antiderivative of f. By the Mean Value Theorem, for every interval $[x_{i-1}, x_i]$ there exists a point c_i in $[x_{i-1}, x_i]$ such that

$$\frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}} = F'(c_i) = f(c_i). \tag{4.1.10}$$

Now $x_i - x_{i-1} = \Delta x_i$, so from (4.1.10) we obtain

$$f(c_i)\Delta x_i = F(x_i) - F(x_{i-1}). \tag{4.1.11}$$

Moreover, $f(x_{i-1}) \leq f(c_i) \leq f(x_i)$, so

$$L(f,P) = \sum_{i=1}^{n} f(x_{i-1})\Delta x_i \leq \sum_{i=1}^{n} f(c_i)\Delta x_i \leq \sum_{i=1}^{n} f(x_i)\Delta x_i = U(f,P). \tag{4.1.12}$$

But, using (4.1.11),

$$\begin{aligned}
\sum_{i=1}^{n} f(c_i)\Delta x_i &= \sum_{i=1}^{n}(F(x_i) - F(x_{i-1})) \\
&= (F(x_1) - F(x_0)) + (F(x_2) - F(x_1)) + (F(x_3) - F(x_2)) + \cdots \\
&\quad\; + (F(x_n) - F(x_{n-1})) \\
&= -F(x_0) + (F(x_1) - F(x_1)) + (F(x_2) - F(x_2)) + (F(x_3) - F(x_3)) + \cdots \\
&\quad\; + (F(x_{n-1}) - F(x_{n-1})) + F(x_n) \\
&= F(x_n) - F(x_0).
\end{aligned}$$

Now $x_0 = 0$ and $x_n = 4$, so

$$F(x_n) - F(x_0) = F(4) - F(0) = 16 - 0 = 16.$$

It now follows from (4.1.11) that, for any partition P,

$$L(f,P) \leq 16 \leq U(f,P). \tag{4.1.13}$$

Since we know that f is integrable on $[0,4]$ (it is continuous on $[0,4]$), the definite integral of f is the only number which satisfies the inequalities in (4.1.13) for any partition P. Hence we must have

$$\int_0^4 2x = 16,$$

in agreement with our geometric argument above.

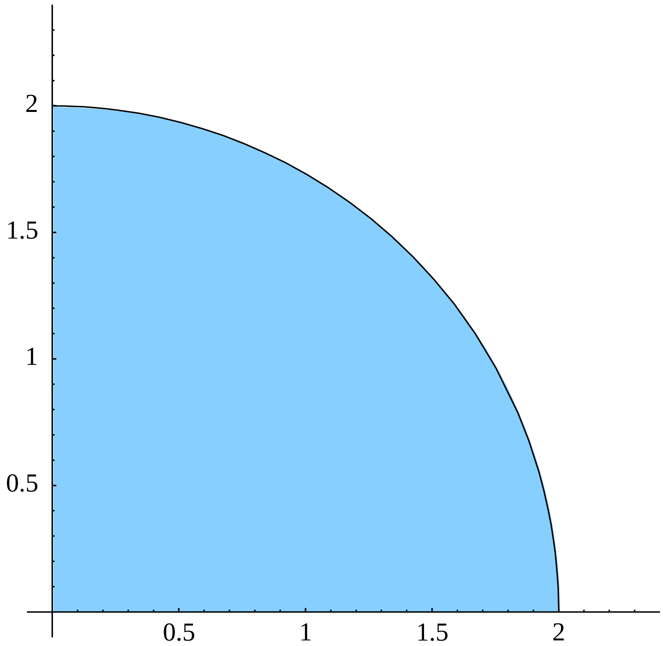

Figure 4.1.6 Region beneath $y = \sqrt{4 - x^2}$ over the interval $[0, 2]$

As we proceed with our study of integration we shall from time to time have occasion to verify that areas computed using a definite integral are consistent with areas computed by other geometric means. At the same time, we shall take this consistency as a given. For example, we shall accept that

$$\int_0^2 \sqrt{4 - x^2}\, dx = \pi,$$

since the region beneath the curve $y = \sqrt{4 - x^2}$ and above the interval $[0, 2]$ is one-quarter of a circle of radius 2 centered at the origin, as shown in Figure 4.1.6.

Example It is important to realize that not all bounded functions are integrable. As an example, consider the function

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is a rational number,} \\ 0, & \text{if } x \text{ is an irrational number.} \end{cases}$$

For example, $f(0.12345) = 1$ and $f\left(\frac{1}{\sqrt{2}}\right) = 0$. Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be a partition of $[0, 1]$. Since every interval $[x_{i-1}, x_i]$, $i = 1, 2, 3, \ldots, n$, contains both rational and irrational numbers, the minimum value of f on $[x_{i-1}, x_i]$ is $m_i = 0$ and the maximum value of f on $[x_{i-1}, x_i]$ is $M_i = 1$. Thus

$$L(f, P) = \sum_{i=1}^{n} m_i \Delta x_i = 0$$

and

$$U(f,P) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} \Delta x_i = 1.$$

Hence any number between 0 and 1 lies between $L(f,P)$ and $U(f,P)$ for any partition P. Since there is not a unique such number, we conclude that f is not integrable on $[0,1]$.

Computing a definite integral directly from the definition is usually a daunting task. We shall take a first look at approximating definite integrals in this section, and then refine these techniques in Section 4.2. In Section 4.3 we will look at the Fundamental Theorem Calculus, a result which will, in certain cases, allow us to compute definite integrals exactly with relative ease.

Riemann sums

Again let f be a function defined on an interval $[a,b]$ and let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be a partition of $[a,b]$. Recall that in the definition of the upper and lower sums, M_i and m_i are chosen, in part, so that $m_i \le f(x) \le M_i$ for all x in $[x_{i-1}, x_i]$, $i = 1, 2, 3, \ldots, n$. It follows that if we choose values $c_1, c_2, c_3, \ldots, c_n$ so that c_i is in the ith interval of the partition (that is, $x_{i-1} \le c_i \le x_i$), then

$$m_i \le f(c_i) \le M_i \tag{4.1.14}$$

for $i = 1, 2, 3, \ldots, n$, and so

$$L(f,P) = \sum_{i=1}^{n} m_i \Delta x_i \le \sum_{i=1}^{n} f(c_i) \Delta x_i \le \sum_{i=1}^{n} M_i \Delta x_i = U(f,P). \tag{4.1.15}$$

If f is integrable, it may be shown that is always possible to choose a partition P so that

$$|U(f,P) - L(f,P)| < \epsilon \tag{4.1.16}$$

for any specified $\epsilon > 0$. It follows that, when f is integrable, it is always possible to find, for any given $\epsilon > 0$, partitions for which

$$\left| \int_a^b f(x)dx - \sum_{i=1}^{n} f(c_i)\Delta x_i \right| < \epsilon \tag{4.1.17}$$

for any choice of the points $c_1, c_2, c_3, \ldots, c_n$. In fact, it may be shown that if we let L be the maximum length of the intervals $[x_{i-1}, x_i]$, then it is possible to find a $\delta > 0$ such that (4.1.17) will hold for any partition with $L < \delta$.

The sum

$$\sum_{i=1}^{n} f(c_i)\Delta x_i \tag{4.1.18}$$

is called a *Riemann sum*, after the German mathematician G. B. F. Riemann (1826-1866). From what we have just seen, we may use Riemann sums to approximate definite integrals. We will consider two important special cases of Riemann sums here (we will look at another in Section 4.2). First, to make calculations simpler, we will restrict to partitions with intervals of equal length. As above, let $P_n = \{x_0, x_1, x_2, \ldots, x_n\}$ be the partition of $[a, b]$ using $n+1$ equally spaced points and let

$$\Delta x = \frac{b-a}{n}$$

be the length of the intervals $[x_{i-1}, x_i]$, $i = 1, 2, 3, \ldots, n$. Note that

$$\lim_{n \to \infty} \Delta x = 0.$$

Hence, if we choose points $c_1, c_2, c_3, \ldots, c_n$ with $x_{i-1} \leq c_i \leq x_i$, then we have, for an integrable f,

$$\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i)\Delta x. \tag{4.1.19}$$

In other words, we may approximate the definite integral $\int_a^b f(x)dx$ to any desired level of accuracy using Riemann sums

$$\sum_{i=1}^{n} f(c_i)\Delta x \tag{4.1.20}$$

with sufficiently large n. To do this efficiently requires specifying how the points c_1, c_2, c_3, $\ldots$, c_n are to be chosen. One method is to simply choose c_i to be the right-hand endpoint of the interval $[x_{i-1}, x_i]$. In that case, since the points in the partition are equally spaced, we have

$$\begin{aligned}
c_1 &= x_1 = x_0 + \Delta x = a + \Delta x, \\
c_2 &= x_2 = x_1 + \Delta x = a + 2\Delta x, \\
c_3 &= x_3 = x_2 + \Delta x = a + 3\Delta x, \\
&\ \ \vdots \\
c_n &= x_n = x_{n-1} + \Delta x = a + n\Delta x.
\end{aligned} \tag{4.1.21}$$

Using these points in (4.1.20), we have

$$\sum_{i=1}^{n} f(c_i)\Delta x = \Delta x \sum_{i=1}^{n} f(a + i\Delta x). \tag{4.1.22}$$

This approximation is known as the *right-hand rule approximation* for $\int_a^b f(x)dx$.

Definition If f is integrable on $[a, b]$, the *right-hand rule approximation* for the definite integral

$$\int_a^b f(x)dx$$

using n intervals is given by

$$A_R = \Delta x \sum_{i=1}^{n} f(a + i\Delta x), \tag{4.1.23}$$

where

$$\Delta x = \frac{b-a}{n}.$$

A similar rule is derived by using the left-hand endpoints of the intervals. In this case we choose

$$\begin{aligned} c_1 &= x_0 = x_0 = a, \\ c_2 &= x_1 = x_0 + \Delta x = a + \Delta x, \\ c_3 &= x_2 = x_1 + \Delta x = a + 2\Delta x, \\ &\vdots \\ c_n &= x_{n-1} = x_{n-2} + \Delta x = a + (n-1)\Delta x. \end{aligned} \tag{4.1.24}$$

Definition If f is integrable on $[a, b]$, the *left-hand rule approximation* for the definite integral

$$\int_a^b f(x)dx$$

using n intervals is given by

$$A_L = \Delta x \sum_{i=0}^{n-1} f(a + i\Delta x), \tag{4.1.25}$$

where

$$\Delta x = \frac{b-a}{n}.$$

Example Returning to our first example, suppose $f(x) = x^2 + 1$ and let A be the area of the region beneath the graph of f and above the interval $[-1, 2]$. With $n = 6$, we have

$$\Delta x = \frac{2-(-1)}{6} = \frac{1}{2}$$

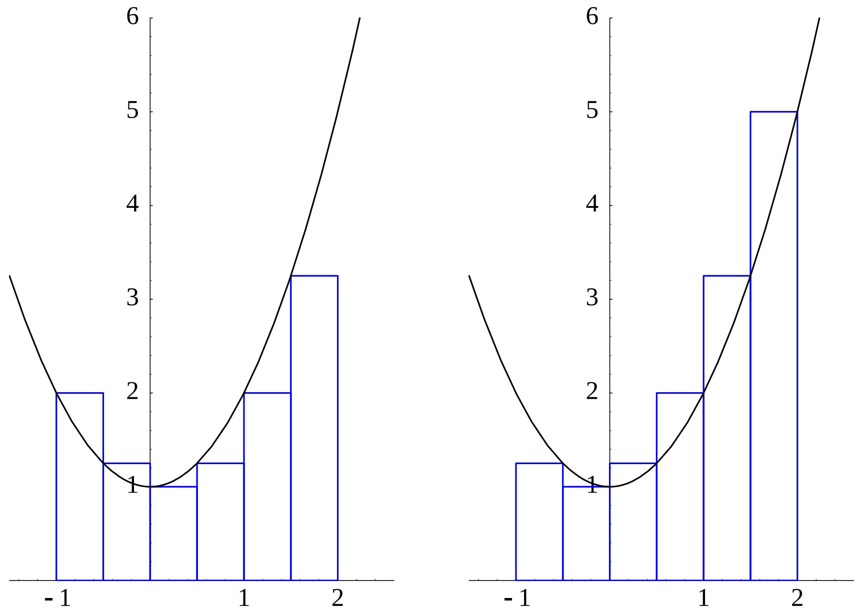

Figure 4.1.7 Left-hand and right-hand rule approximations for $\int_{-1}^{2}(x^2+1)dx$

and the left-hand rule approximation for A is

$$\begin{aligned}
A_L &= \frac{1}{2}\sum_{i=0}^{5} f\left(-1+\frac{1}{2}i\right)\\
&= \frac{1}{2}\left(f(-1)+f\left(-\frac{1}{2}\right)+f(0)+f\left(\frac{1}{2}\right)+f(1)+f\left(\frac{3}{2}\right)\right)\\
&= \frac{1}{2}\left(2+\frac{5}{4}+1+\frac{5}{4}+2+\frac{13}{4}\right)\\
&= \frac{1}{2}\left(\frac{43}{4}\right)\\
&= \frac{43}{8} = 5.375.
\end{aligned}$$

See the figure on the left in Figure 4.1.7. Similarly, the right-hand rule approximation is

$$\begin{aligned}
A_R &= \frac{1}{2}\sum_{i=1}^{6} f\left(-1+\frac{1}{2}i\right)\\
&= \frac{1}{2}\left(f\left(-\frac{1}{2}\right)+f(0)+f\left(\frac{1}{2}\right)+f(1)+f\left(\frac{3}{2}\right)+f(2)\right)\\
&= \frac{1}{2}\left(\frac{5}{4}+1+\frac{5}{4}+2+\frac{13}{4}+5\right)
\end{aligned}$$

$$= \frac{1}{2}\left(\frac{55}{4}\right)$$
$$= \frac{55}{8} = 6.875.$$

See the figure on the right in Figure 4.1.7. Recall that, for a partition of 6 intervals of equal length, we computed a lower sum of 4.875 and an upper sum 7.375. Hence, as we would expect for any Riemann sums, A_L and A_R lie between the lower and upper sums.

Using $n = 100$ and a computer, we find $A_L = 5.955$ and $A_R = 6.045$, which again lie between the lower sum of 5.925 and the upper sum of 6.075.

Example Now let A be the area of the region beneath the graph of

$$g(t) = \frac{1}{t}$$

over the interval $[1, 10]$, as shown in Figure 4.1.8. Then

$$A = \int_1^{10} \frac{1}{t}\, dt.$$

In Section 6.2 we will see that this integral is equal to the natural logarithm of 10, which, to 6 decimal places, is 2.302585. The following table summarizes the left-hand and right-hand rule approximations for A:

n	A_R	A_L	$\lvert A - A_R\rvert$	$\lvert A - A_L\rvert$
10	1.960214	2.770214	0.342371	0.467629
20	2.116477	2.521477	0.186108	0.218892
40	2.205491	2.407991	0.097094	0.105406
80	2.253003	2.354253	0.049582	0.051668
160	2.277534	2.328159	0.025052	0.025574
320	2.289994	2.315307	0.012591	0.012722

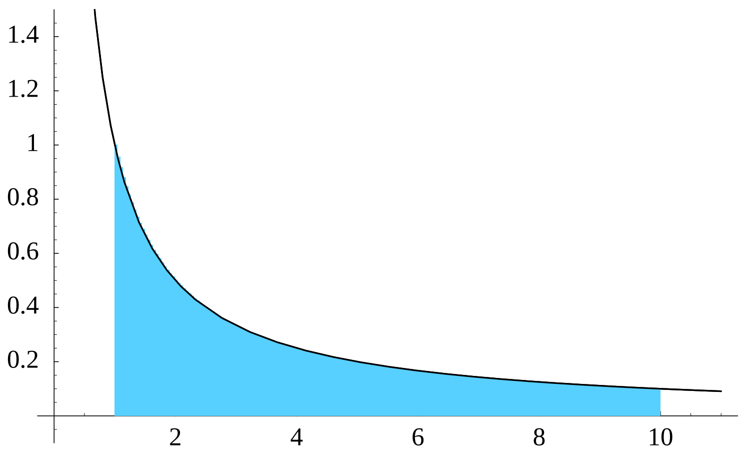

Figure 4.1.8 Region beneath $y = \dfrac{1}{t}$ over the interval $[1, 10]$

As we should expect, the error in our approximations decreases as the number of subdivisions increases. What is more interesting to note is that, in this particular case, when the number of subdivisions is doubled, the error committed by both the right-hand and the left-hand rules decreases by a factor of, roughly, $\frac{1}{2}$. For example, this might lead us to predict that the error in using 640 intervals would be about 0.0063; in fact, it turns out to be 0.006312 for the right-hand rule and 0.006344 for the left-hand rule. This type of behavior is typical for this method of approximation, a point we will come back to when we investigate other methods of approximation in Section 4.2.

Properties of the definite integral

Since the integral of an integrable function may be computed as the limit of Riemann sums, the basic properties of limits and sums hold true for integrals as well. In particular, if f and g are integrable on $[a, b]$ and k is any constant, then

$$\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx, \tag{4.1.26}$$

$$\int_a^b (f(x) - g(x))dx = \int_a^b f(x)dx - \int_a^b g(x)dx, \tag{4.1.27}$$

and

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx. \tag{4.1.28}$$

Example We know that

$$\int_0^3 xdx = \frac{9}{2}$$

(since the region under the graph of $y = x$ over the interval $[0, 3]$ is a triangle with base of length 3 and a height of 3) and

$$\int_0^3 4dx = 12$$

(either using (4.1.9) or the fact that the region under the graph of $y = 4$ is a rectangle with base of length 3 and a height of 4), so it follows from (4.1.24) that

$$\int_0^3 (x+4)dx = \int_0^3 xdx + \int_0^3 4dx = \frac{9}{2} + 12 = \frac{33}{2}.$$

Example The graph of $g(t) = \sqrt{1-t^2}$ over the interval $[-1, 1]$ is a semicircle of radius 1 centered at the origin, so

$$\int_{-1}^1 5\sqrt{1-t^2}\ dt = 5\int_{-1}^1 \sqrt{1-t^2}\ dt = \frac{5\pi}{2}.$$

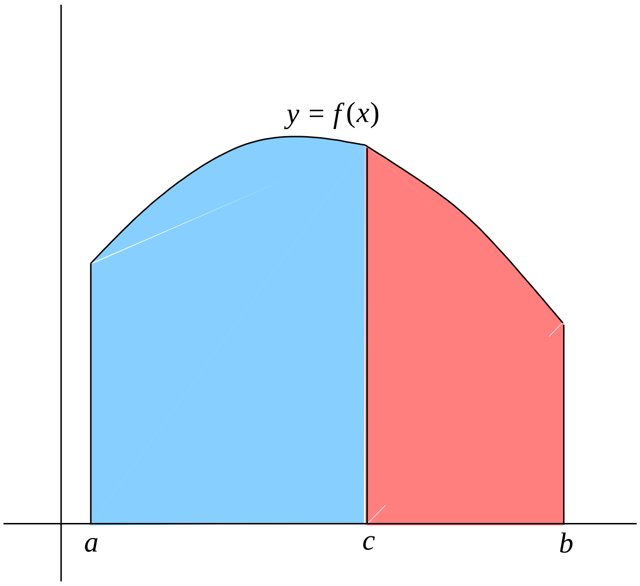

Figure 4.1.9 $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

Now suppose f is integrable on $[a, b]$ and c is a point with $a < c < b$. It may be shown that f is integrable on both $[a, c]$ and $[c, b]$. Moveover, using partitions which include c, we may write a Riemann sum for f over $[a, b]$ as the sum of two Riemann sums, the first over the interval $[a, c]$ and the second over the interval $[c, b]$. After taking limits, it follows that

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx. \tag{4.1.29}$$

If $f(x) \geq 0$ for all x in $[a, b]$, we may think of (4.1.29) as saying that the area under the graph of f over the interval $[a, b]$ is equal to the area under the graph of f over the interval $[a, c]$ plus the area under the graph of f over the interval $[c, b]$. See Figure 4.1.9.

Example Suppose

$$f(x) = \begin{cases} x, & \text{if } 0 \leq x \leq 1, \\ 3, & \text{if } 1 < x \leq 2. \end{cases}$$

The region under the graph of f is shown in Figure 4.1.10. Now

$$\int_0^2 f(x)dx = \int_0^1 f(x)dx + \int_1^2 f(x)dx = \int_0^1 xdx + \int_1^2 3dx = \frac{1}{2} + 3 = \frac{7}{2}.$$

Technically, before applying (4.1.29) in the previous example we should have verified that f is integrable on $[0, 2]$. Since f is not continuous on $[0, 2]$, its integrability does not follow from our previous results. However, f is an example of what is known as a *piecewise continuous* function, which we will now define.

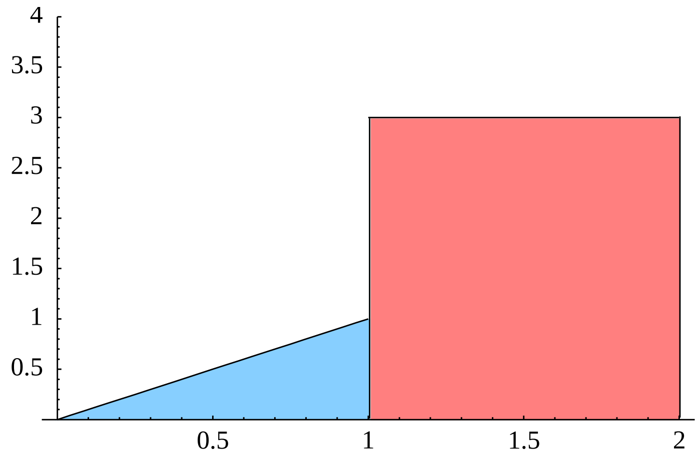

Figure 4.1.10 $\int_0^2 f(x)dx = \int_0^1 f(x)dx + \int_1^2 f(x)dx$

Definition A function is said to be *piecewise continuous* on an interval $[a, b]$ if there is a partition $P = \{x_0, x_1, x_2, \ldots, x_n\}$ of $[a, b]$ such that f is continuous on each open interval (x_{i-1}, x_i), $i = 1, 2, 3, \ldots, n$; has limits from both the right and the left at each partition point x_i, $i = 1, 2, 3, \ldots, x_{n-1}$; and has a right-hand limit at a and a left-hand limit at b.

Proposition If f is piecewise continuous on $[a, b]$, then f is integrable on $[a, b]$.

Example The function f in the previous example is piecewise continuous on $[0, 2]$, and hence integrable on $[0, 2]$ by the previous proposition.

Now suppose f and g are both integrable on $[a, b]$ and $f(x) \leq g(x)$ for all x in $[a, b]$. It follows that for any given partition P, the upper sum of g will be greater than or equal to the corresponding upper sum of f. Since the definite integral is the largest number less than or equal to the value of any upper sum, it follows that

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx. \tag{4.1.30}$$

Example Since $0 \leq x^2 \leq x$ for all x in $[0, 1]$, we have

$$\int_0^1 0dx \leq \int_0^1 x^2dx \leq \int_0^1 xdx.$$

Now

$$\int_0^1 0dx = 0(1-0) = 0$$

and

$$\int_0^1 xdx = \frac{1}{2},$$

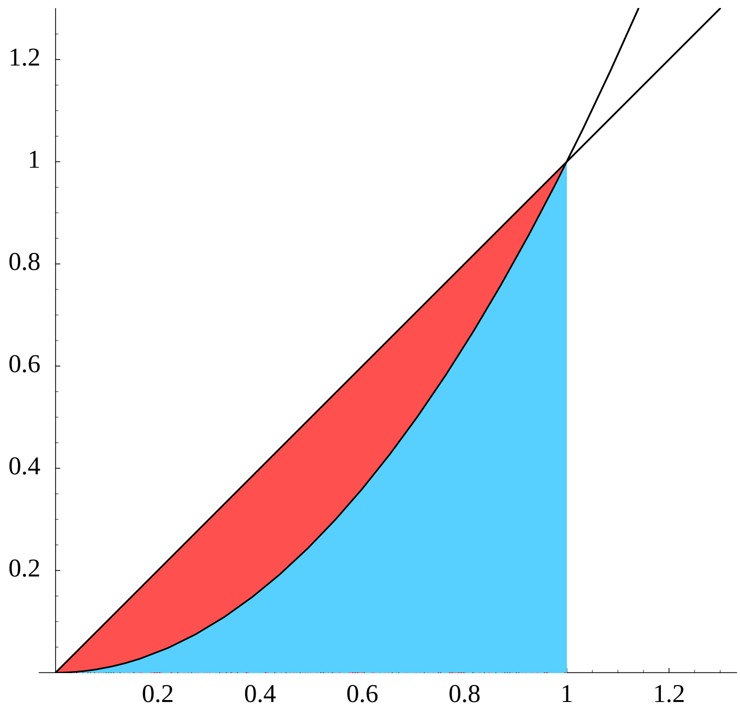

Figure 4.1.11 $0 \leq \int_0^1 x^2 dx \leq \int_0^1 x dx$

so it follows that

$$0 \leq \int_0^1 x^2 \leq \frac{1}{2}.$$

See Figure 4.1.11.

Geometric interpretations

The original motivation for this section was the problem of finding the area of a region in the plane. Given an integrable function f with $f(x) \geq 0$ for all x in an interval $[a, b]$, we eventually defined the area of the region beneath the graph of f and above the interval $[a, b]$ to be $\int_a^b f(x)dx$. Now suppose $f(x) \leq 0$ for all x in $[a, b]$ and let R be the region between the graph of f and the interval $[a, b]$. If S is the region beneath the graph of $y = -f(x)$ and above $[a, b]$, then we have

$$\text{area of } R = \text{ area of } S = \int_a^b -f(x)dx = -\int_a^b f(x)dx. \tag{4.1.31}$$

Hence, in this case, $\int_a^b f(x)dx$ is not the area of the region R, but rather

$$\int_a^b f(x)dx = -(\text{area of } R). \tag{4.1.32}$$

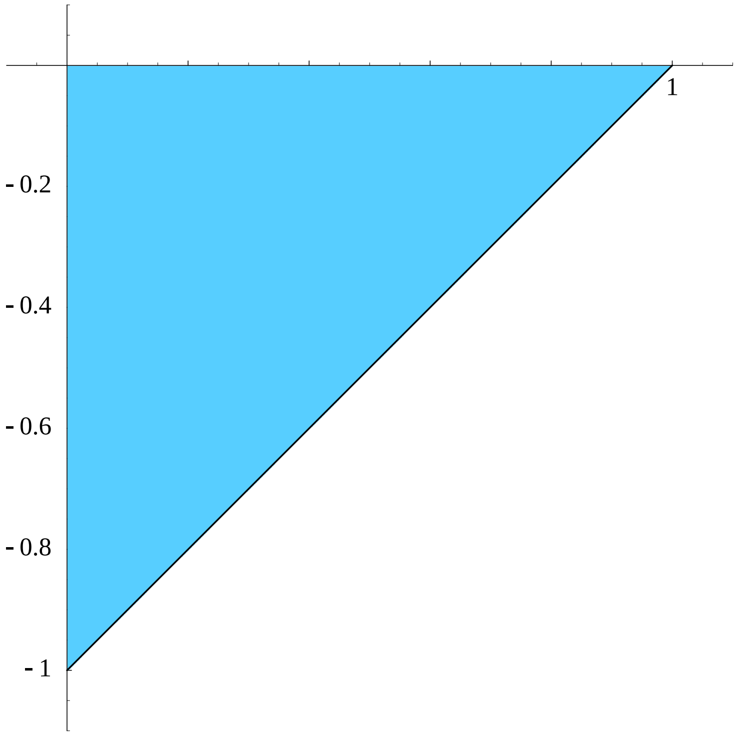

Figure 4.1.12 Region between the graph of $y = x - 1$ and the interval $[0, 1]$

Example If $f(x) = x - 1$, then $f(x) \leq 0$ for all x in $[0, 1]$. Since the region between the graph of f and the interval $[0, 1]$ on the x-axis is a triangle with area $\frac{1}{2}$, we must have

$$\int_0^1 (x - 1)dx = -\frac{1}{2}.$$

See Figure 4.1.12.

More generally, we may think of $\int_a^b f(x)dx$ as representing the difference of the area of any regions between the graph of f and the x-axis which lie above the x-axis and the area of those regions which lie below the x-axis. For example, we have

$$\int_{-\pi}^{\pi} \sin(x)dx = 0$$

because the area of the region beneath the graph of $y = \sin(x)$ over the interval $[0, \pi]$ is negated by the area of the region between the graph $y = \sin(x)$ and the interval $[-\pi, 0]$, as shown in Figure 4.1.13.

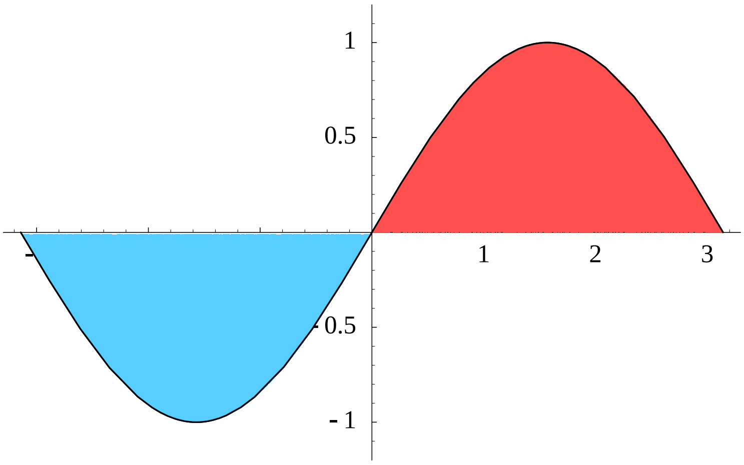

Figure 4.1.13 Area above the x-axis cancels area beneath the x-axis

Problems

1. For each of the following, find upper and lower bounds for the area of the region beneath the given curve over the given interval using four inscribed rectangles and four circumscribed rectangles.

   (a) $y = \dfrac{1}{x}$ on $[1, 5]$ (b) $y = x^2$ on $[0, 4]$

   (c) $y = x^2 + 1$ on $[-2, 2]$ (d) $y = \sin(x)$ on $[0, \pi]$

2. Find the upper and lower sums for the following integrals using a partition with six equal intervals.

   (a) $\displaystyle\int_1^4 3x\,dx$ (b) $\displaystyle\int_{-2}^4 x^2 dx$

   (c) $\displaystyle\int_{-\pi}^{\pi} \cos(x)dx$ (d) $\displaystyle\int_{-2}^{2} (4 - x^2)dx$

   (e) $\displaystyle\int_0^1 (x^3 - x)dx$ (f) $\displaystyle\int_0^1 \sin(2\pi t)dt$

3. For each of the following, approximate the area beneath the graph of the function over the given interval using the right-hand and left-hand rules with four intervals.

   (a) $f(x) = x^2$ on $[0, 4]$ (b) $f(x) = x^2$ on $[-2, 2]$

   (c) $g(t) = \dfrac{1}{t}$ on $[1, 9]$ (d) $g(t) = \dfrac{1}{t}$ on $[1, 2]$

   (e) $h(x) = x^3$ on $[0, 1]$ (f) $f(t) = 1 - t^2$ on $[-1, 1]$

4. For each of the following, approximate the area beneath the graph of the function over the given interval using the right-hand and left-hand rules with 100 intervals.

   (a) $f(x) = x^2$ on $[0, 1]$ (b) $g(x) = \sin(x)$ on $[0, \pi]$

   (c) $f(t) = t^3$ on $[0, 2]$ (d) $g(z) = z^2$ on $[-2, 2]$

(e) $h(x) = \dfrac{1}{x}$ on $[1, 2]$

(f) $f(x) = \sqrt{1 - x^2}$ on $[-1, 1]$

(g) $h(\theta) = \sec(\theta)$ on $\left[-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right]$

(h) $g(t) = \sin(2t)$ on $\left[0, \dfrac{\pi}{2}\right]$

5. Use the right-hand and left-hand rules with four intervals to approximate the following definite integrals.

(a) $\displaystyle\int_0^2 x^2 dx$

(b) $\displaystyle\int_2^3 \frac{1}{x}\, dx$

(c) $\displaystyle\int_{-\pi}^{\pi} \cos(t) dt$

(d) $\displaystyle\int_{-3}^{1} s^3 ds$

(e) $\displaystyle\int_{-1}^{1} (x^2 - 1) dx$

(f) $\displaystyle\int_{-\pi}^{\pi} \sin(z) dz$

6. Use the right-hand and left-hand rules with 100 intervals to approximate the following definite integrals.

(a) $\displaystyle\int_0^3 x^2 dx$

(b) $\displaystyle\int_{-1}^{2} x^3 dx$

(c) $\displaystyle\int_{-2}^{2} \sqrt{4 - t^2}\, dt$

(d) $\displaystyle\int_0^{2\pi} \sin(x) dx$

(e) $\displaystyle\int_{-1}^{1} (x^2 - 1) dx$

(f) $\displaystyle\int_0^{\pi} \sin(3\theta) d\theta$

(g) $\displaystyle\int_{-1}^{0} \frac{x}{x^2 + 1}\, dx$

(h) $\displaystyle\int_{-4}^{-2} \frac{1}{t}\, dt$

7. Use geometric arguments to determine the value of each of the following definite integrals.

(a) $\displaystyle\int_0^4 x dx$

(b) $\displaystyle\int_0^3 (2x + 3) dx$

(c) $\displaystyle\int_0^3 \sqrt{9 - x^2}\, dx$

(d) $\displaystyle\int_{-2}^{2} 4\sqrt{4 - t^2}\, dt$

(e) $\displaystyle\int_{-2}^{2} x^3 dx$

(f) $\displaystyle\int_0^{2\pi} \sin(t) dt$

8. Suppose

$$f(x) = \begin{cases} x + 1, & \text{if } 0 \le x \le 1, \\ 4, & \text{if } 1 < x \le 3. \end{cases}$$

Combine geometric arguments with properties of definite integrals to determine the value of the following definite integrals.

(a) $\displaystyle\int_0^1 f(x) dx$

(b) $\displaystyle\int_1^3 f(x) dx$

(c) $\displaystyle\int_0^3 f(x)dx$ (d) $\displaystyle\int_0^2 f(x)dx$

9. The definition of $\displaystyle\int_a^b f(x)dx$ assumes $a < b$.
   (a) Explain why it would be reasonable to define
   $$\int_a^a f(x)dx = 0.$$
   (b) Explain why it would be reasonable to define
   $$\int_a^b f(x)dx = -\int_b^a f(x)dx$$
   whenever $a > b$.
   (c) Using the definitions given in (a) and (b), and assuming that f is integrable on the appropriate intervals, show that
   $$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$
   whether $a \leq c \leq b$, $a \leq b \leq c$, $b \leq a \leq c$, $b \leq c \leq a$, $c \leq a \leq b$, or $c \leq b \leq a$. Note that this generalizes (4.1.29).

10. Suppose f is integrable on $[a, b]$ and m and M are constants such that $m \leq f(x) \leq M$ for all x in $[a, b]$. Show that
    $$m(b - a) \leq \int_a^b f(x)dx \leq M(b - a).$$

11. Given that f is integrable on $[a, b]$, it may be shown that $g(x) = |f(x)|$ is also integrable on $[a, b]$. Show that
    $$\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|dx.$$
    Hint: Use the fact that $-|f(x)| \leq f(x) \leq |f(x)|$ for all x in $[a, b]$.

12. In this section we showed that
    $$\int_0^4 2xdx = 16$$
    directly from the definition of the definite integral (with some help from the Mean Value Theorem).

(a) Use these ideas to show that

$$\int_0^1 x dx = \frac{1}{2}.$$

(b) More generally, show that

$$\int_0^b x dx = \frac{b^2}{2}.$$

(c) Let

$$F(x) = \int_0^x t dt.$$

What is the relationship between F and the function $f(x) = x$?

13. In this section we showed that

$$\int_0^4 2x = 16$$

directly from the definition of the definite integral (with some help from the Mean Value Theorem). Use these ideas to show that

$$\int_0^1 x^2 dx = \frac{1}{3}.$$