

***Difference Equations  
to  
Differential Equations***

**Section 3.6  
Newton's Method**

Many problems in mathematics involve, at some point or another, solving an equation for an unknown quantity. An equation of the form  $f(x) = 0$  may be solved for  $x$  by simple algebra if  $f$  is an affine function and by the quadratic formula if  $f$  is a quadratic polynomial. There are formulas similar to the quadratic formula for both cubic and quartic polynomials, but they are, in general, very cumbersome. One of the most interesting results of mathematics, due to Niels Henrik Abel (1802-1829), is that there does not exist an analogue of the quadratic formula for quintic polynomials. For this and other reasons, it turns out that in many situations solving an equation  $f(x) = 0$  for  $x$  requires using a method which can approximate the solutions to a predetermined level of accuracy.

In Section 2.5 we discussed one such method, the bisection algorithm, for approximating the solutions of an equation. The strong point of the bisection algorithm is that, once an appropriate starting interval has been found, the method will always find a solution to any desired level of accuracy; its weakness lies in the slowness with which the successive approximations approach the solution. In this section we will discuss another method, known as *Newton's method*, for approximating solutions to an equation. In distinction to the bisection algorithm, Newton's method does not always work, but, when it does, it is in general remarkably fast.

Suppose we wish to find a solution to the equation  $f(x) = 0$  for a given function  $f$ . Recall that, geometrically, this corresponds to finding the point where the curve  $y = f(x)$  crosses the  $x$ -axis. To start Newton's method, we must first have an initial guess  $x_0$ . Frequently, we find the initial guess by graphing the curve  $y = f(x)$  and letting  $x_0$  be a point close to where the curve crosses the  $x$ -axis. Given the initial guess  $x_0$ , let  $T_0$  be the best affine approximation to  $f$  at  $x_0$ . That is, define  $T_0$  by

$$T_0(x) = f'(x_0)(x - x_0) + f(x_0). \quad (3.6.1)$$

The idea behind Newton's method is to obtain an improved estimate of a solution to  $f(x) = 0$  by replacing the equation  $f(x) = 0$  with the simpler equation  $T_0(x) = 0$ . If we let  $x_1$  denote the solution to the latter equation, then we have  $T_0(x_1) = 0$ , that is

$$f'(x_0)(x_1 - x_0) + f(x_0), \quad (3.6.2)$$

from which it follows that

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}. \quad (3.6.3)$$

Geometrically,  $x_1$  is the point at which the line tangent to  $f$  at  $(x_0, f(x_0))$  crosses the  $x$ -axis, as shown in Figure 3.6.1. To improve upon this approximation, we solve the equation

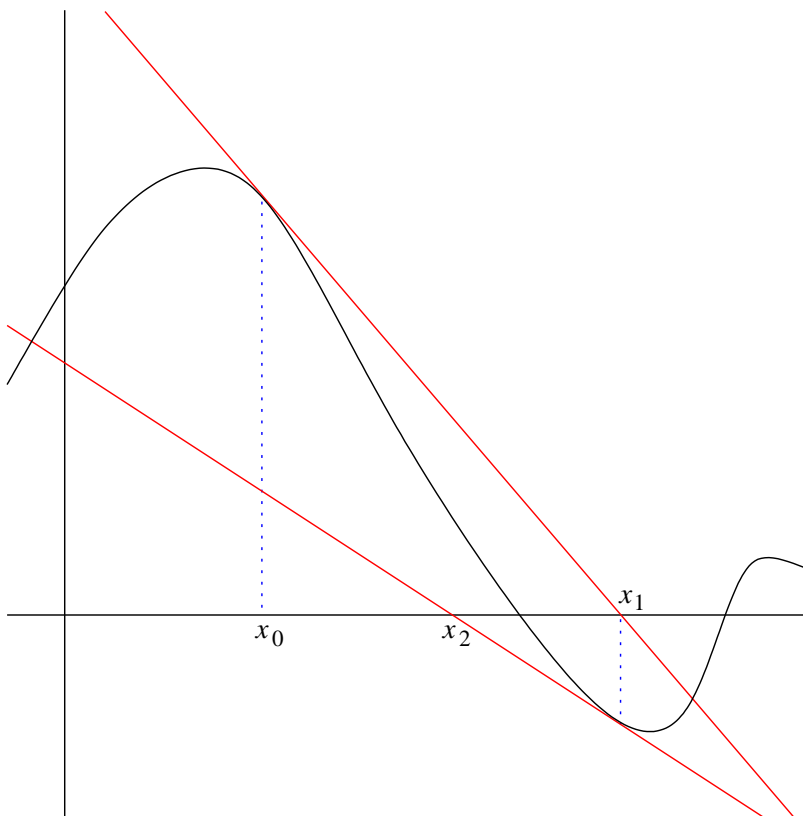


Figure 3.6.1 Two iterations of Newton's method

$T_1(x) = 0$ , where  $T_1$  is the best affine approximation to  $f$  at  $x_1$ . If we let  $x_1$  denote the solution to this equation, then

$$f'(x_1)(x_2 - x_1) + f(x_1) = 0, \quad (3.6.4)$$

and so

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}. \quad (3.6.5)$$

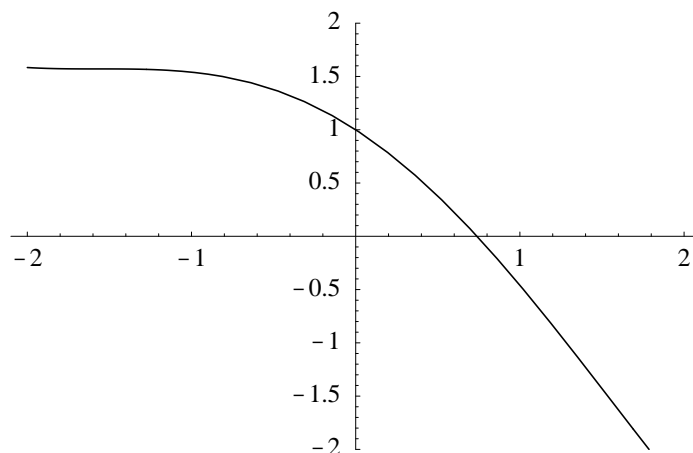
We continue in this manner to generate a sequence of approximations  $x_0, x_1, x_2, x_3, \dots$ , until we reach the desired degree of accuracy. Specifically, if we have found  $x_n$ , we find  $x_{n+1}$  by solving the equation  $T_n(x) = 0$ , where  $T_n$  is the best affine approximation to  $f$  at  $x_n$ . Hence we have

$$f(x_n)(x_{n+1} - x_n) + f(x_n) = 0, \quad (3.6.6)$$

which implies

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \quad (3.6.7)$$

In other words, beginning with an initial guess  $x_0$ , Newton's method generates a sequence  $\{x_n\}$  using the difference equation (3.6.7). In most cases (although certainly not all), if  $x_0$  is a good initial guess,  $\lim_{n \rightarrow \infty} x_n = r$ , where  $r$  is a solution of  $f(x) = 0$ , that is,  $f(r) = 0$ .

Figure 3.6.2 Graph of  $f(x) = \cos(x) - x$ 

In any practical case we need to know when to stop generating successive approximations using (3.6.7). Since we do not know the exact solutions to the equation (if we did, we would not be using Newton's method to start with), we can never know for sure how far a given approximation is from a solution. What is done in practice is to generate terms until the difference between successive terms is less than a predetermined tolerance level. That is, if we decide that we want our approximation to be off by no more than  $\epsilon$ , then we stop when  $|x_{n+1} - x_n| < \epsilon$ .

**Newton's method** To approximate a solution to an equation  $f(x) = 0$  to within a tolerance of  $\epsilon$  beginning with an initial guess  $x_0$ , compute the sequence of approximations  $x_0, x_1, x_2, x_3, \dots$ , using the difference equation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad (3.6.8)$$

stopping when  $|x_{n+1} - x_n| < \epsilon$ .

**Example** Suppose we wish to find a solution to the equation  $\cos(x) = x$  with an error of no more than 0.0001. Then we should let  $f(x) = \cos(x) - x$  and look for solutions to  $f(x) = 0$ . Since  $\cos(x)$  will always be between  $-1$  and  $1$ , we know that any solution to  $\cos(x) = x$  must lie in the interval  $[-1, 1]$ . Moreover, from the graph of  $f$  in Figure 3.6.2, we can see that the equation  $f(x) = 0$  has only one solution, and this solution lies between  $0$  and  $1$ . Alternatively, we could note that

$$f(0) = 1 > 0$$

and

$$f(1) = \cos(1) - 1 < 0,$$

which imply, by the Intermediate Value Theorem, that there is a solution in the interval  $[0, 1]$ . In either case, we will use  $x_0 = 0.5$  for our initial guess. Now

$$f'(x) = -\sin(x) - 1,$$

so

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.5 - \frac{f(0.5)}{f'(0.5)} = 0.755222,$$

where we have rounded the result to 6 decimal places. Substituting this back into (3.6.8) gives us

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.752222 - \frac{f(0.755222)}{f'(0.755222)} = 0.739142.$$

Since

$$|x_2 - x_1| = 0.016080 > 0.0001,$$

we continue and compute

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.739142 - \frac{f(0.739142)}{f'(0.739142)} = 0.739085.$$

Now we have

$$|x_3 - x_2| = 0.000057 < 0.0001,$$

so we stop and use 0.7391 as our approximation to the solution of  $\cos(x) = x$ . For comparison, with the bisection algorithm starting from the initial interval  $[0, 1]$ , we would have had to iterate 13 times before obtaining an approximation to the solution with an error less than 0.0001.

**Example** As an example of where Newton's method goes wrong, consider the equation  $f(x) = 0$ , where

$$f(x) = \frac{x}{1+x^2}.$$

Clearly,  $x = 0$  is the only solution to this equation. However, beginning with an initial guess  $x_0 = 0.75$ , Newton's method yields the following sequence (where we have rounded each value to 5 digits):

$$x_0 = 0.75000$$

$$x_1 = -1.9286$$

$$x_2 = -5.2756$$

$$x_3 = -10.944$$

$$x_4 = -22.073$$

$$x_5 = -44.237$$

$$x_6 = -88.518$$

$$x_7 = -177.06$$

$$x_8 = -354.13$$

$$x_9 = -708.27$$

$$x_{10} = -1416.5$$

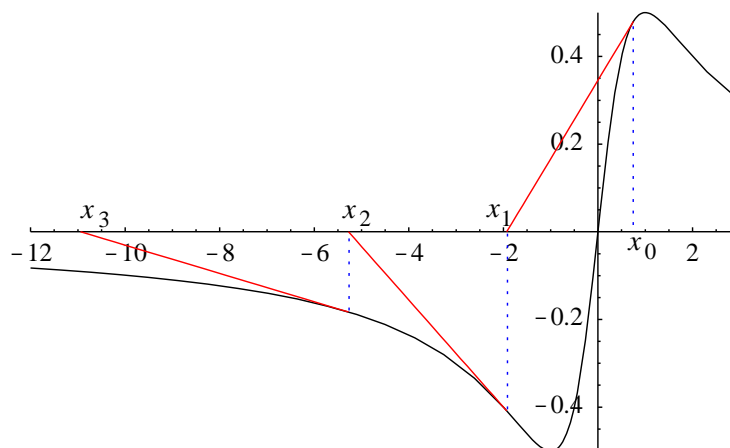


Figure 3.6.3 Newton's method diverging from a solution

Instead of converging to the solution at 0, this sequence seems to be diverging toward  $-\infty$ . In fact, geometrically this appears to be exactly the case, as can be seen in Figure 3.6.3. Here the problem comes from the fact that the graph of  $f$  approaches 0 asymptotically as  $x$  goes to  $-\infty$ . Newton's method is following the curve as it approaches 0, as it should, but, since there is no solution in this direction, the result is that the iterates are getting farther and farther away from the solution at  $x = 0$ .

Note that in the last example,

$$f'(x) = \frac{1 - x^2}{(1 + x^2)^2},$$

giving us  $f'(1) = 0$ . Hence if we had started with an initial guess of  $x_0 = 1$ , an application of the difference equation (3.6.8) would require a division by 0, which, of course, cannot be done. Geometrically, the tangent line to the graph of  $f$  at  $(1, 0.5)$  is horizontal, and hence never crosses the  $x$ -axis, implying that there are no solutions to the equation  $T(x) = 0$  if  $T$  is the best affine approximation to  $f$  at 1. Thus we must avoid starting Newton's method at a point where the derivative is 0.

## Problems

- For each of the following equations, use a graph to obtain initial guesses for solutions and then apply Newton's method to locate the solutions within 0.0001.
  - $x^5 - 6x^3 + 2x = 2$
  - $\sin(x) = x^2$
  - $\cos(t) = t^2$
  - $\cos^2(t) - t^2 = 0$
  - $2\sin(x) = \sqrt{x+1}$
  - $6x^4 - 12x^3 + 4x - 1 = 0$
- Even though we know that for a positive number  $c$  the equation  $x^2 - c = 0$  has the exact solutions  $-\sqrt{c}$  and  $\sqrt{c}$ , we may use Newton's methods to find decimal approximations to these square roots.

- (a) Show that the sequence  $x_0, x_1, x_2, \dots$ , of Newton's method approximations to a solution of  $x^2 - c = 0$  satisfies the difference equation

$$x_{n+1} = \frac{x_n + \frac{c}{x_n}}{2}$$

for  $n = 0, 1, 2, \dots$

- (b) Use the difference equation from (a) to approximate  $\sqrt{2}$ ,  $\sqrt{3}$ , and  $\sqrt{11}$  with an error of less than 0.00001.
- (c) Can you see an intuitive reason why, starting with a positive initial guess, the sequence defined by the difference equation in part (a) might converge to  $\sqrt{c}$ ?
- (d) Assuming that  $L = \lim_{n \rightarrow \infty} x_n$  for the sequence defined in (a), show that either  $L = -\sqrt{c}$  or  $L = \sqrt{c}$ .
3. Use Newton's method to approximate  $\sqrt[3]{2}$  with an error less than 0.00001.
4. Use Newton's method to approximate  $\sqrt[3]{5}$  with an error less than 0.00001.
5. The method outlined in Problem 2 for approximating square roots was known to the Greeks and perhaps to the Babylonians. For an account of this and other aspects of Babylonian algebra, read Chapter 3 of *Mathematics in Civilization* by H. L. Resnikoff and R. O. Wells, Jr. (Dover Publications, Inc., New York, 1984).
6. What happens when you apply Newton's method to find solutions to the equation

$$x^3 - 5x = 0$$

starting with an initial guess of  $x_0 = 1$ ? Explain this geometrically with a graph.

7. We know that when solving an equation  $f(x) = 0$  using Newton's method, different initial guesses may lead to different solutions, and some may not converge to a solution at all. The problem of determining which initial guesses converge to a specified solution is surprisingly complicated, involving what mathematicians call *fractals*. For an account of this phenomenon, read pages 217-220 of *Chaos* by James Gleick (Viking Penguin, Inc., New York, 1987). Also, see the picture on the sixth color plate following page 114 in the same book.