Difference Equations
to
Differential Equations

Section 3.4

Differentiation of Compositions of Functions

In this section we will consider the relationship between the derivative of the composition of two functions and the derivatives of the individual functions being composed. We shall see that the resulting differentiation rule, known as the *chain rule*, will be useful in a variety of situations in our later work. The following example will set the stage.

Example Consider a spherical balloon which is being inflated so that its radius is increasing at a rate of 2 centimeters per second. If we let r denote the radius of the balloon in centimeters, t denote time in seconds, and V denote the volume of the balloon in cubic centimeters, then we know that $r = 2t$ and

$$V = \frac{4}{3}\pi r^3.$$

Moreover, we can see that, as a function of t,

$$V = \frac{4}{3}\pi(2t)^3 = \frac{32}{3}\pi t^3.$$

At time $t = 5$, the rate of change of the radius with respect to time is

$$\left.\frac{dr}{dt}\right|_{t=5} = 2 \text{ centimeters per second},$$

the rate of change of the volume with respect to the radius is

$$\left.\frac{dV}{dr}\right|_{r=10} = \left.4\pi r^2\right|_{r=10} = 400\pi \text{ centimeters per centimeter},$$

and the rate of change of the volume with respect to time is

$$\left.\frac{dV}{dt}\right|_{t=5} = \left.32\pi t^2\right|_{t=5} = 800\pi \text{ cubic centimeters per second},$$

where $\frac{dV}{dr}$ is evaluated at $r = 10$ since this is the value of r when $t = 5$. It follows that

$$\left.\frac{dV}{dt}\right|_{t=5} = \left.\frac{dV}{dr}\right|_{r=10} \left.\frac{dr}{dt}\right|_{t=5}.$$

That is, the overall rate of change of V with respect to t is the product of the rate of change of V with respect to r and the rate of change of r with respect to t. This is an example of the chain rule. Viewed in this manner, the chain rule is saying that if V changes 400π times as fast as r and r changes 2 times as fast as t, then V changes $(400\pi)(2) = 800\pi$ times as fast as t.

Another interesting special case of the chain rule arises with the composition of two affine functions. Specifically, if $f(x) = ax + b$ and $g(x) = cx + d$, where a, b, c, and d are all constants, then

$$f \circ g(x) = f(g(x)) = f(cx + d) = a(cx + d) + b = acx + ad + b.$$

Thus the slope of graph of $f \circ g$ is ac, the product of the slopes of the graphs of f and g. In terms of derivatives, this says that

$$(f \circ g)'(x) = ac = f'(g(x))g'(x).$$

The chain rule says this relationship holds for all differentiable functions.

For the general case, suppose g is differentiable at a point c and f is differentiable at $g(c)$. We wish to compute the value of the derivative of $f \circ g$ at c. We have

$$(f \circ g)'(c) = \lim_{h \to 0} \frac{f \circ g(c+h) - f \circ g(c)}{h} = \lim_{h \to 0} \frac{f(g(c+h)) - f(g(c))}{h}. \tag{3.4.1}$$

As with our demonstrations of the quotient and product rules, we need to manipulate (3.4.1) into a form which allows us to evaluate the limit in terms of what we already know. The trick that works this time is to multiply and divide by $g(c+h) - g(c)$. However, we must be aware of one possible complication: In order to divide by $g(c+h) - g(c)$ we must be assured that $g(c+h) - g(c) \neq 0$ for all h in some interval about 0. We will assume that this is the case. If in fact this were not the case, then one can show that both $(f \circ g)'(c) = 0$ and $g'(c) = 0$, giving us the desired result that

$$(f \circ g)'(c) = f'(g(c))g'(c).$$

With our assumption, we have

$$(f \circ g)'(c) = \lim_{h \to 0} \left(\frac{f(g(c+h)) - f(g(c))}{g(c+h) - g(c)} \right) \left(\frac{g(c+h) - g(c)}{h} \right). \tag{3.4.2}$$

Since g is differentiable at c, we have

$$\lim_{h \to 0} \frac{g(c+h) - g(c)}{h} = g'(c). \tag{3.4.3}$$

Since f is differentiable at $g(c)$, if we let $s = g(c+h) - g(c)$, then

$$\lim_{h \to 0} \frac{f(g(c+h)) - f(g(c))}{g(c+h) - g(c)} = \lim_{s \to 0} \frac{f(g(c) + s) - f(g(c))}{s} = f'(g(c)), \tag{3.4.4}$$

where we have used the continuity of g at c to ascertain that s goes to 0 as h goes to 0. Putting (3.4.2), (3.4.3) and (3.4.4) together, we now have

$$(f \circ g)'(c) = f'(g(c))g'(c), \tag{3.4.5}$$

which is our desired result.

Chain Rule If f and g are differentiable, then

$$(f \circ g)'(x) = f'(g(x))g'(x). \tag{3.4.6}$$

Example Suppose $h(x) = (1 + x^2)^{10}$. Then $h(x) = f \circ g(x)$ where $g(x) = 1 + x^2$ and $f(x) = x^{10}$. Now

$$g'(x) = 2x$$

and

$$f'(x) = 10x^9,$$

so

$$h'(x) = (f \circ g)'(x) = f'(g(x))g'(x) = f'(1 + x^2)(2x) = 10(1 + x^2)^9(2x) = 20x(1 + x^2)^9.$$

Note that the preceding example is a particular case of the following general example. If g is a differentiable function, $n \neq 0$ is an integer, and $h(x) = (g(x))^n$, then $h(x) = f \circ g(x)$ where $f(x) = x^n$. Then we have

$$f'(x) = nx^{n-1},$$

and so

$$h'(x) = (f \circ g)'(x) = f'(g(x))g'(x) = n(g(x))^{n-1}g'(x).$$

That is,

$$\frac{d}{dx}(g(x)^n) = n(g(x))^{n-1}g'(x). \tag{3.4.7}$$

Example To illustrate the previous comments,

$$\frac{d}{dx}(3x - 2)^6 = 6(3x - 2)^5 \frac{d}{dx}(3x - 2) = 6(3x - 2)^5(3) = 18(3x - 2)^5.$$

Example For another illustration, if

$$f(x) = \frac{3}{(x^3 + 4)^5},$$

then

$$f'(x) = (-5)(3)(x^3 + 4)^{-6} \frac{d}{dx}(x^3 + 4) = -15(x^3 + 4)^{-6}(3x^2) = -\frac{45x^2}{(x^3 + 4)^6}.$$

If we translate the chain rule into the notation of Leibniz, we obtain a formulation like that of the first example. Specifically, if we let $y = f(x)$ and $x = g(t)$, then

$$\left.\frac{dy}{dt}\right|_{t=c} = (f \circ g)'(c) = f'(g(c))g'(c) = \left.\frac{dy}{dx}\right|_{x=g(c)} \left.\frac{dx}{dt}\right|_{t=c}. \tag{3.4.8}$$

For short, we write

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}. \tag{3.4.9}$$

This formula is easy to remember, but at the same time care must be taken to remember that if we want to evaluate $\frac{dy}{dt}$ at $t = c$, then we must evaluate $\frac{dy}{dx}$ at $x = g(c)$.

Example Suppose that for a certain city, when the population of the city is p, the total amount of waste deposited in the city landfill every day is given by $W = 5\sqrt{p}$ pounds per day. Moreover, suppose that the population of the city is growing so that t years from now the population will be

$$p = 100,000(1 + 0.04t + 0.008t^2).$$

To find the rate of change of W with respect to t five years from now, we note that $p = 140,000$ when $t = 5$ and then compute

$$\left.\frac{dW}{dp}\right|_{p=140,000} = \left.\frac{5}{2\sqrt{p}}\right|_{p=140,000} = \frac{5}{2\sqrt{140,000}}$$

and

$$\left.\frac{dp}{dt}\right|_{t=5} = 100,000(0.04 + 0.016t)\big|_{t=5} = 12,000.$$

Hence the rate of increase of the number of pounds of waste in the landfill after five years is, in pounds per day per year,

$$\left.\frac{dW}{dt}\right|_{t=5} = \left.\frac{dW}{dp}\right|_{p=140,000} \left.\frac{dp}{dt}\right|_{t=5} = \left(\frac{5}{2\sqrt{140,000}}\right)(12,000) = \frac{30,000}{\sqrt{140,000}} = 80.12,$$

where the final answer is rounded to 2 decimal places.

Differentiation of algebraic functions

At this point the only thing keeping us from routinely differentiating any algebraic function is that we do not have a rule for handling exponents which are rational numbers, but not integers. We now consider this problem. Suppose $y = x^n$, where $n = \frac{p}{q}$ for nonzero integers p and q. Then

$$y^q = \left(x^{\frac{p}{q}}\right)^q = x^p. \tag{3.4.10}$$

Differentiating the left-hand side of (3.4.9) with respect to x gives us

$$\frac{d}{dx}y^q = qy^{q-1}\frac{dy}{dx}, \tag{3.4.11}$$

where the factor $\frac{dy}{dx}$ is a consequence of the special case of the chain rule in (3.4.7). Of course,

$$\frac{d}{dx}x^p = px^{p-1}. \tag{3.4.12}$$

We may equate (3.4.11) and (3.4.12) (by (3.4.10) they are the derivatives of equal functions) to obtain

$$qy^{q-1}\frac{dy}{dx} = px^{p-1}. \tag{3.4.13}$$

Solving for $\frac{dy}{dx}$, we have

$$\frac{dy}{dx} = \frac{px^{p-1}}{qy^{q-1}} = \frac{p}{q}x^{p-1}y^{1-q}. \tag{3.4.14}$$

Recalling that $y = x^{\frac{p}{q}}$ and $n = \frac{p}{q}$, (3.4.14) becomes

$$\frac{dy}{dx} = \frac{p}{q}x^{p-1}\left(x^{\frac{p}{q}}\right)^{1-q} = \frac{p}{q}x^{p-1}x^{\frac{p}{q}-p} = \frac{p}{q}x^{\frac{p}{q}-1} = nx^{n-1}. \tag{3.4.15}$$

Hence we may now state the following proposition as an extension of our previous results.

Proposition If $n \neq 0$ is a rational number, then

$$\frac{d}{dx}x^n = nx^{n-1}. \tag{3.4.16}$$

Example We have

$$\frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{\frac{1}{2}} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}},$$

in agreement with our result in Section 3.2.

Example If

$$f(x) = \frac{3}{\sqrt{x^2+1}},$$

then

$$f'(x) = \frac{d}{dx}3(x^2+1)^{-\frac{1}{2}} = -\frac{3}{2}(x^2+1)^{-\frac{3}{2}}(2x) = -\frac{3x}{(x^2+1)^{\frac{3}{2}}}.$$

Implicit differentiation

The technique used in the demonstration of the last proposition is of general use. Any equation involving two variables, such as $f(x, y) = 0$, determines a curve in the plane consisting of the set of all ordered pairs (x, y) which satisfy the equation. Such a curve need not be the graph of a function. For example, the curve associated with $x^2+y^2-25 = 0$, or, more simply, $x^2 + y^2 = 25$, is a circle of radius 5 centered at the origin, which is not the graph of any function. However, for a specified point on the curve, it may be the case that a segment of the curve containing that point is the graph of some function; hence the

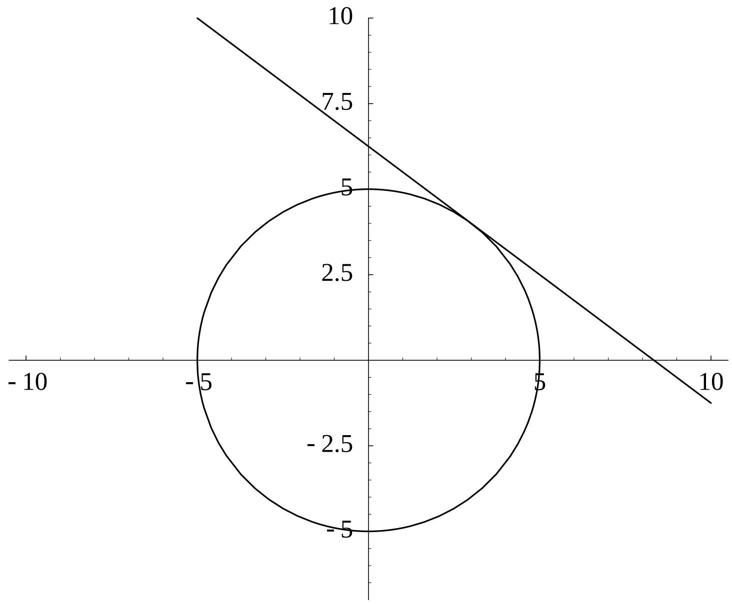

Figure 3.4.1 Tangent line to the circle $x^2 + y^2 = 25$ at $(3, 4)$

curve may have a tangent line at this point. For example, $(3, 4)$ is a point on the curve $x^2 + y^2 = 25$ which lies on the half of the circle lying above the x-axis and, considered by itself, this piece of the circle is the graph of a function, namely, the function $y = \sqrt{25 - x^2}$. To find the slope of the tangent line at such a point on the curve, we may borrow the technique we used in demonstrating the previous proposition. That is, we differentiate both sides of the equation, treating one variable as a function of the other. If we treat y as a function of x, then, differentiating with respect to x and using the chain rule, we obtain an equation involving $\frac{dy}{dx}$ which we can then solve for $\frac{dy}{dx}$. For the equation $x^2 + y^2 = 25$, we have

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}25.$$

Since

$$\frac{d}{dx}(x^2 + y^2) = 2x + 2y\frac{dy}{dx}$$

and

$$\frac{d}{dx}25 = 0,$$

we have

$$2x + 2y\frac{dy}{dx} = 0.$$

Solving for $\frac{dy}{dx}$, we have

$$\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$$

at all points (x, y) for which $y \neq 0$. Now we have

$$\left.\frac{dy}{dx}\right|_{(x,y)=(3,4)} = -\frac{3}{4},$$

and so the equation of the tangent line at $(3,4)$ is

$$y = -\frac{3}{4}(x-3) + 4.$$

The circle with equation $x^2 + y^2 = 25$ and the tangent line at $(3,4)$ are shown in Figure 3.4.1. Note that our procedure would not work to find the tangent lines to the circle at $(-5,0)$ and $(5,0)$. However, the tangents lines at these points are vertical, and, hence, do not have a slope. Although it is beyond the scope of this book to provide a justification, it is in fact the case that the technique outlined in this example will work to find the slope of the tangent line at all points on the curve that have a tangent line with a slope.

This technique for finding derivatives is called *implicit differentiation* because we did not use an explicit formula for y in terms x. In this case we could have obtained the same result by first solving for y in terms of x for values close to $(3,4)$, giving us $y = \sqrt{25-x^2}$, and then evaluating the derivative of this function at $x = 3$. However, this is not always possible or desirable; in many cases implicit differentiation is significantly simpler even if an explicit solution is possible.

Example Consider the problem of finding the best affine approximation to the curve with equation

$$y^3 + 3xy^2 - xy + x = 7$$

near the point $(2,1)$. To find $\frac{dy}{dx}$, we compute

$$\frac{d}{dx}(y^3 + 3xy^2 - xy + x) = \frac{d}{dx}7,$$

which give us

$$\frac{d}{dx}y^3 + 3x\frac{d}{dx}y^2 + 3y^2\frac{d}{dx}x - \left(x\frac{dy}{dx} + y\frac{d}{dx}x\right) + \frac{d}{dx}x = 0.$$

Computing the derivatives on the left-hand side gives us

$$3y^2\frac{dy}{dx} + 3x\left(2y\frac{dy}{dx}\right) + 3y^2(1) - x\frac{dy}{dx} - y(1) + 1 = 0.$$

Hence

$$3y^2\frac{dy}{dx} + 6xy\frac{dy}{dx} + 3y^2 - x\frac{dy}{dx} - y + 1 = 0,$$

from which it follows that

$$\frac{dy}{dx}(3y^2 + 6xy - x) = y - 3y^2 - 1.$$

Solving for $\frac{dy}{dx}$, we have

$$\frac{dy}{dx} = \frac{y - 3y^2 - 1}{3y^2 + 6xy - x},$$

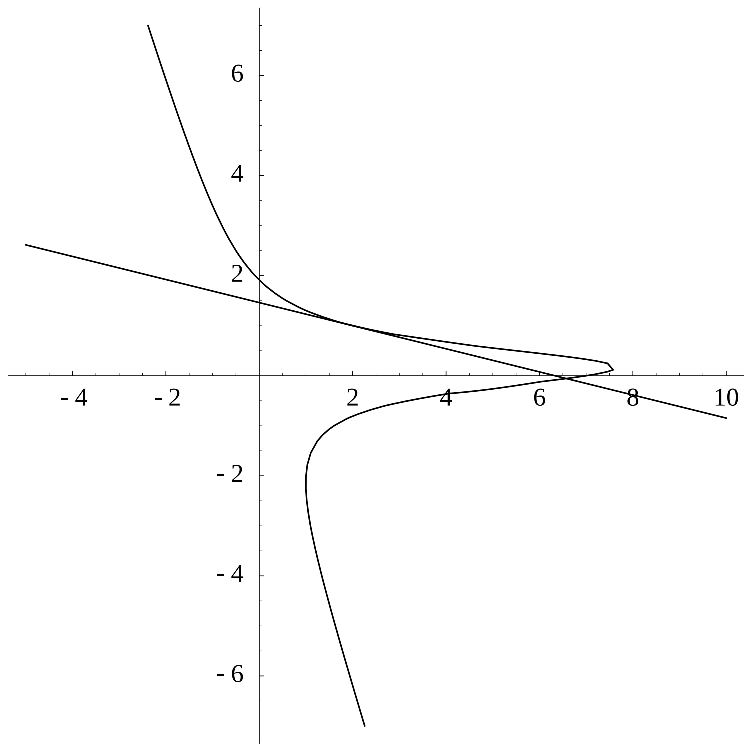

Figure 3.4.2 Curve with equation $y^3 + 3xy^2 - xy + x = 7$ and tangent line at $(2, 1)$

which holds at all points for which the denominator is not 0. Thus

$$\left.\frac{dy}{dx}\right|_{(x,y)=(2,1)} = \frac{1-3-1}{3+12-2} = -\frac{3}{13}.$$

So the best affine approximation at $(2, 1)$ is given by

$$T(x) = -\frac{3}{13}(x-2) + 1.$$

The equation in this example does not specify y as a function of x (in fact, in Figure 3.4.2 we can see that there are at least two other values of y that correspond to $x = 2$), but there is a segment of the curve through $(2, 1)$ which is the graph of some function. For this function, which we have not explicitly found, T is the best affine approximation at $x = 2$. For example, if we denote this unknown function by h, we know that

$$h(2.05) \approx T(2.05) = -\frac{3}{13}(0.05) + 1 = 0.9885,$$

where we have rounded the result to four decimal places. Put another way, the point $(2.05, 0.9885)$ is an approximate solution to the equation

$$y^3 + 3xy^2 - xy + x = 7.$$

At this point we can routinely find the derivative of any algebraic function. In the next section we will consider the derivatives of the trigonometric functions.

Problems

1. Find the derivative of each of the following functions.
   (a) $f(x) = (4x+5)^4$
   (b) $g(x) = 13x(x^2+2)^5$
   (c) $h(t) = \dfrac{3}{2(6t-2)^2}$
   (d) $f(s) = \dfrac{3s-4}{(s^3+2)^4}$
   (e) $g(z) = (3z+4)^3(2z^2+z)^2$
   (f) $f(x) = \dfrac{(3x+4)^3(8x-13)^4}{(2x+3)}$

2. For each of the following, find the derivative of the dependent variable with respect to the independent variable.
   (a) $s = 4t^2(t^2-1)^2$
   (b) $z = -\dfrac{s^2(4s-3)^2}{s^2+1}$
   (c) $q = \sqrt{3t^3-4t}$
   (d) $y = \dfrac{3x}{\sqrt{3x+4}}$
   (e) $x = 8t(4t+5)^{-2}$
   (f) $u = 3(v^2+4)^{-\frac{2}{3}}$
   (g) $y = (3x-1)^{\frac{1}{5}}$
   (h) $v = \sqrt{u^2+(3u-2)^2}$

3. Find the best affine approximation to the function
$$f(x) = \frac{3x}{(x^2+1)^2}$$
   at $x = 2$.

4. (a) Find the best affine approximation to $f(x) = (1+x)^h$ at $x = 0$, where $h \neq 0$ is a constant.
   (b) Use your result from (a) to approximate $\sqrt{1.06}$ and compare with the value obtained from a calculator.
   (c) Use your result from (a) to approximate $\sqrt[3]{1.06}$ and compare with the value obtained from a calculator.
   (d) Use your result from (a) to approximate $\sqrt[5]{1.06}$ and compare with the value obtained from a calculator.

5. Find the equation of the line tangent to each of the following curves at the indicated point.
   (a) $x^2+3y^2 = 21$ at $(3,2)$
   (b) $x^2-3y^2 = 4$ at $(4,2)$
   (c) $x^2+3xy+y^2 = 11$ at $(2,1)$
   (d) $y^5+2x^2y^2-x^2 = 10$ at $(3,1)$
   (e) $x^5+xy+y^5 = 3$ at $(1,1)$
   (f) $4x^2-3xy-2xy^2 = 26$ at $(-2,1)$

6. Suppose values for $f(x)$, $f'(x)$, $g(x)$, and $g'(x)$ are as given in the following table.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	1	2	2	3
1	2	-1	0	2
2	0	3	1	-2

Find $k'(0)$ for each of the following.

(a) $k(x) = f \circ g(x)$
(b) $k(x) = g \circ f(x)$
(c) $k(x) = g \circ g(x)$
(d) $k(x) = f \circ f(x)$
(e) $k(x) = (f \circ g) \circ f(x)$
(f) $k(x) = g(f(x))f(x)$

7. Show that if $g'(c) = 0$, then $(g \circ g)'(c) = 0$.

8. Suppose the sides of cube are increasing at a rate of 3 centimeters per minute. At what rate is the volume of the cube increasing when the length of one of the sides is 10 centimeters?

9. A pebble is dropped in a pond of water. Suppose that the resulting circular wave has a radius given by $r = 20\sqrt{t}$ centimeters after t seconds. Find the rate of change of the area of the wave with respect to time after 5 seconds.

10. The volume of a balloon is increasing at a rate of 50 cubic centimeters per second. At what rate is the radius increasing when the radius is 10 centimeters?

11. The kinetic energy of an object moving in a straight line is given by

$$K = \frac{1}{2}mv^2,$$

where m is the mass of the object and v is its velocity. If the acceleration of the object, $a = \frac{dv}{dt}$, is a constant 9.8 meters per second per second, find $\frac{dK}{dt}$ when $v = 10$ meters per second.

12. Ship A passes a buoy at 10:00 a.m. and heads north at 20 miles per hour. Ship B passes the same buoy at 11:00 a.m. and heads east at 25 miles per hour. If s is the distance between the ships, what is $\frac{ds}{dt}$ at noon?

13. Suppose the height of a rectangle is growing at a rate of 0.1 inches per second while its length is growing at a rate of 0.2 inches per second. When the height of the rectangle is 4 inches and its length is 8 inches, at what rate is the area of the rectangle increasing?

14. The work force of a certain factory is growing at rate of 2 per month while the average productivity of a worker is growing at a rate of 4 units per month. If the work force is currently 100 and the average productivity per month is 200 units, at what rate is the total productivity per month of the factory increasing?

15. (a) What happens in Problem 14 if the work force is declining by 2 per month?
(b) What happens in Problem 14 if the average productivity is decreasing by 5 per month?

16. A circular oil slick is 0.03 feet thick and has a radius which is increasing at a rate of 2 feet per hour. When the radius is 100 feet, at what rate is the volume of the oil slick increasing?

17. Oil is being added to a circular oil slick at the rate of 100 cubic feet per minute. If the oil slick is 0.05 feet thick, at what rate is the radius of the oil slick increasing when the radius is 400 feet?

18. In Section 2.2 we mentioned that the period of a pendulum of length b centimeters undergoing small oscillations is given by

$$T = 2\pi\sqrt{\frac{b}{g}} \text{ seconds,}$$

where $g = 980$ centimeters per second per second. Suppose the length of the pendulum changes as a function of temperature τ so that

$$\frac{db}{d\tau} = 0.08 \text{ centimeters per degree Celsius.}$$

(a) Find $\frac{dT}{d\tau}$ when $b = 20$ centimeters.

(b) Use (a) to approximate the effect on T of a 1°C increase in temperature. Do the same for a 2°C increase and a 2°C decrease.