Section 2.4

Continuous Functions

Given the work of the previous section, we are now in a position to state a clear definition of the notion of continuity. We will have several related definitions, but the fundamental definition is that of continuity at a point. Intuitively, continuity at a point c for a function f means that the values of f for points near c do not change abruptly from the value of f at c. Section 2.3 has shown that, mathematically, this means that as x approaches c, the value of $f(x)$ must be approaching $f(c)$. Hence we have the following basic definition.

Definition We say that a function f is *continuous at a point* c if

$$\lim_{x \to c} f(x) = f(c). \tag{2.4.1}$$

It is important to note that this definition places three conditions on the behavior of the function f near the point c. Namely, f is continuous at the point c if (1) f is defined at c, (2) $\lim\limits_{x \to c} f(x)$ exists, and (3) $\lim\limits_{x \to c} f(x) = f(c)$.

Corresponding to one-sided limits, we have the notions of continuity from the left and from the right.

Definition We say that a function f is *continuous from the left at a point* c if

$$\lim_{x \to c^-} f(x) = f(c). \tag{2.4.2}$$

We say that a function f is *continuous from the right at a point* c if

$$\lim_{x \to c^+} f(x) = f(c). \tag{2.4.3}$$

Simply to say that a function f is continuous, without specifying some particular point, means that the function is continuous, in the proper sense, at all points where it is defined. Here "in the proper sense" means, for example, that if f is defined only on a closed interval $[a, b]$, then we cannot ask for continuity at a or b, since it is possible to discuss only one-sided limits at these points, but it is possible to inquire about continuity from the right at a and continuity from the left at b.

Definition We say a function f is *continuous on the open interval* (a, b) if f is continuous at every point in (a, b). We say f is *continuous on the closed interval* $[a, b]$ if f is continuous on (a, b), continuous from the right at a, and continuous form the left at b.

In the previous section we saw that if f and g are polynomials and c is a point with $g(c) \neq 0$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)}.$$

The following proposition restates this fact in terms of our new definitions.

Proposition If h is a rational function and h is defined at the point c, then h is continuous at c. In particular, if h is a polynomial, then h is continuous on the entire real line $(-\infty, \infty)$.

This theorem gives us a very large class of functions which we know to be continuous. As we progress, we will add many more types of functions to this class.

Example Consider the function $f(x) = 3x^3 - 6x + 3$. Since f is a polynomial, it is continuous on $(-\infty, \infty)$. That is, for any real number c, f is continuous at c.

Example Consider

$$g(t) = \frac{8t - 13t^2}{3t - 4}.$$

Then g is a rational function, and so is continuous at all points in its domain. That is, g is continuous for all real numbers c except $c = \frac{4}{3}$. Put another way, g is continuous on the intervals $(-\infty, \frac{4}{3})$ and $(\frac{4}{3}, \infty)$.

Example Suppose

$$h(z) = \begin{cases} z^2 - 2, & \text{if } z \leq 1, \\ 4z - 2, & \text{if } z > 1. \end{cases}$$

On the interval $(-\infty, 1]$, h is a polynomial; thus h is continuous on $(-\infty, 1]$. Note that this does not necessarily mean that h is continuous at 1, only that h is continuous from the left at 1. Similarly, on the interval $(1, \infty)$, h is a polynomial and hence is continuous on $(1, \infty)$. To check for continuity at 1, we note that

$$\lim_{z \to 1^-} h(z) = \lim_{z \to 1^-} (z^2 - 2) = -1,$$

while

$$\lim_{z \to 1^+} h(z) = \lim_{z \to 1^+} (4z - 2) = 2.$$

Since these limits are different, we know that $\lim\limits_{z \to 1} h(z)$ does not exist. Thus h is not continuous at 1. As we saw in Section 2.3, this behavior results in a break in the graph of h at $z = 1$. See Figure 2.4.1.

Example Suppose

$$f(s) = \begin{cases} s + 1, & \text{if } s < 0, \\ s^2 + 1, & \text{if } s \geq 0. \end{cases}$$

Similar to the situation in the previous example, f is continuous on the intervals $(-\infty, 0)$ and $[0, \infty)$ since it is a polynomial on both of these intervals. Now

$$\lim_{s \to 0^-} f(s) = \lim_{s \to 0^-} (s + 1) = 1$$

and

$$\lim_{s \to 0^+} f(s) = \lim_{s \to 0^+} (s^2 + 1) = 1.$$

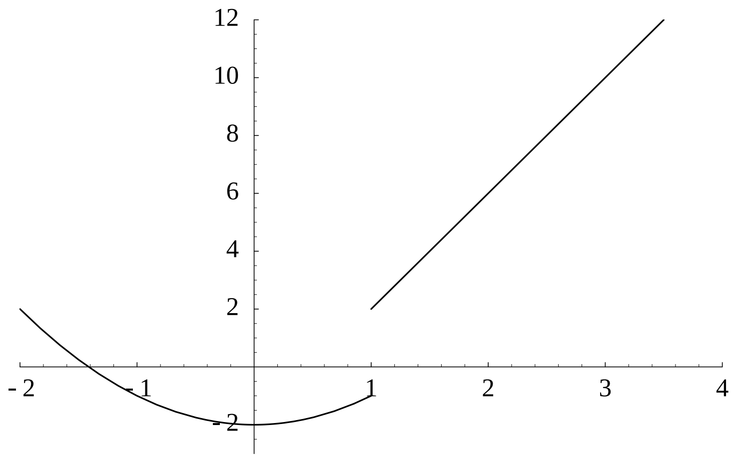

Figure 2.4.1 Graph of $y = h(z)$

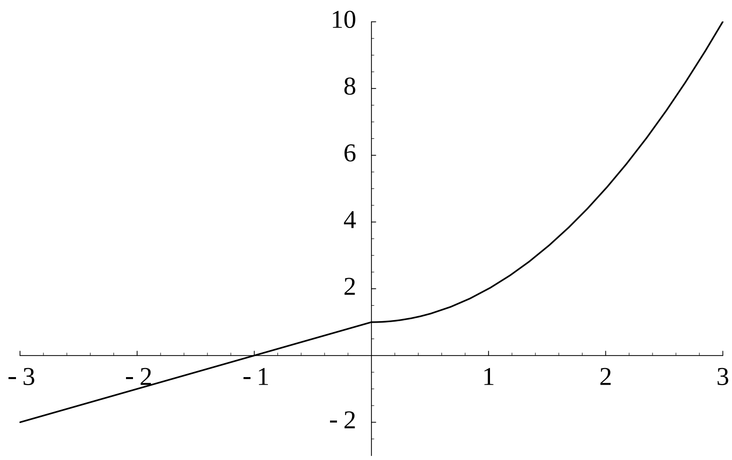

Figure 2.4.2 Graph of $y = f(s)$

Thus $\lim_{s \to 0} f(s) = 1$; since $f(0) = 1$, we have

$$\lim_{s \to 0} f(s) = 1 = f(0).$$

Thus f is continuous at 0. Altogether this shows that f is continuous on the entire interval $(-\infty, \infty)$. As we see in Figure 2.4.2, the graph of f does not have a break at $s = 0$.

Example Suppose

$$g(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & \text{if } x \neq 2, \\ 6, & \text{if } x = 2. \end{cases}$$

Then, since g is a rational function on the intervals $(-\infty, 2)$ and $(2, \infty)$, and is defined throughout these intervals, g is continuous on the intervals $(-\infty, 2)$ and $(2, \infty)$. To check

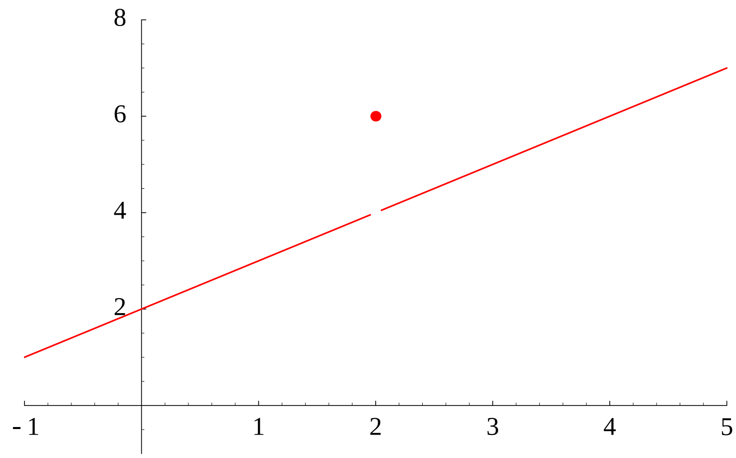

Figure 2.4.3 Graph of $y = g(x)$

for continuity at 2, we notice that

$$\lim_{x\to 2} g(x) = \lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2}(x+2) = 4,$$

while $g(2) = 6$. Hence $\lim\limits_{x\to 2} g(x) \neq g(2)$, and so g is not continuous at 2. See Figure 2.4.3.

It is interesting to note that in the last example the function g could be made continuous if its value at 2 were changed from 6 to 4. In general, if, for a function f and a point c, $\lim\limits_{x\to c} f(x) = L$, but f is not continuous at c because either f is not defined at c or $f(c) \neq L$, we can define a new function h such that $h(x) = f(x)$ for all $x \neq c$ and h is continuous at c. Namely, if we let

$$h(x) = \begin{cases} f(x), & \text{if } x \neq c, \\ L, & \text{if } x = c, \end{cases}$$

then $h(x) = f(x)$ for all $x \neq c$ and

$$\lim_{x\to c} h(x) = \lim_{x\to c} f(x) = L = h(c).$$

Thus h is a function which is identical to f everywhere except at c, but, unlike f, is continuous at c. In this case we say that f has a *removable discontinuity* at c. Note that the existence of a limit at c is essential in order for a discontinuity at c to be removable.

The following proposition lists some properties of continuous functions, all of which are consequences of our results about limits in Section 2.3.

Proposition Suppose the functions f and g are both continuous at a point c and k is a constant. Then the functions which take on the following values for a variable x are also continuous at c:

$$kf(x), \tag{2.4.4}$$

$$f(x) + g(x), \tag{2.4.5}$$

$$f(x) - g(x), \tag{2.4.6}$$

$$f(x)g(x), \tag{2.4.7}$$

$$\frac{f(x)}{g(x)}, \tag{2.4.8}$$

provided $g(c) \neq 0$, and

$$(f(x))^p, \tag{2.4.9}$$

provided p is a rational number and $(f(x))^p$ is defined on an open interval containing c.

Example It follows from (2.4.9) that functions of the form $f(x) = x^p$, where p is a rational number, are continuous throughout their domain. For example, $f(x) = \sqrt{x}$ is continuous on $[0, \infty)$.

Example Using (2.4.8) and (2.4.9),

$$g(t) = \frac{\sqrt{3t+2}}{2t}$$

is continuous for all points t where $3t + 2 \geq 0$ and $t \neq 0$. Thus g is continuous on the intervals $[-\frac{2}{3}, 0)$ and $(0, \infty)$.

At this point we have the tools necessary to determine questions of continuity for algebraic functions. We will now show that the sine and cosine functions are continuous on $(-\infty, \infty)$. For $0 < x < \frac{\pi}{2}$, consider the point $C = (\cos(x), \sin(x))$ on the unit circle centered at the origin. If we let $A = (0, 0)$ and $B = (1, 0)$, as in Figure 2.4.4, then the area of $\triangle ABC$ is

$$\frac{1}{2}\sin(x).$$

The area of the sector of the circle cut off by the arc from B to C is the fraction $\frac{x}{2\pi}$ of the area of the entire circle; hence, this area is

$$\frac{x}{2\pi}\pi = \frac{x}{2}.$$

Since this sector contains $\triangle ABC$, we have

$$0 < \frac{1}{2}\sin(x) < \frac{x}{2},$$

from which it follows that

$$0 < \sin(x) < x.$$

Since

$$\lim_{x \to 0^+} x = 0,$$

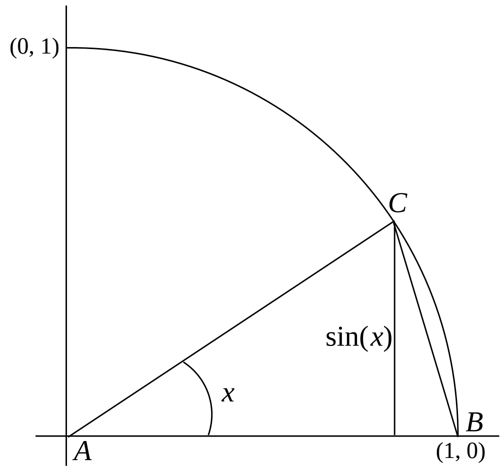

Figure 2.4.4

it follows that

$$\lim_{x \to 0^+} \sin(x) = 0.$$

Moreover, we also have

$$\lim_{x \to 0^-} \sin(x) = \lim_{x \to 0^+} \sin(-x) = -\lim_{x \to 0^+} \sin(x) = 0,$$

so

$$\lim_{x \to 0} \sin(x) = 0. \tag{2.4.10}$$

Since $\sin(0) = 0$, this shows that sine is continuous at 0. Now for $-\frac{\pi}{2} < x < \frac{\pi}{2}$,

$$\cos(x) = \sqrt{1 - \sin^2(x)}.$$

Hence

$$\lim_{x \to 0} \cos(x) = \lim_{x \to 0} \sqrt{1 - \sin^2(x)} = \sqrt{1 - \lim_{x \to 0} \sin^2(x)} = 1. \tag{2.4.11}$$

Since $\cos(0) = 1$, this shows that cosine is continuous at 0. For an arbitrary c, we have, using the angle addition formulas for sine and cosine,

$$\begin{aligned}
\lim_{x \to c} \sin(x) &= \lim_{h \to 0} \sin(c + h) \\
&= \lim_{h \to 0} (\sin(c)\cos(h) + \cos(c)\sin(h)) \\
&= \sin(c) \lim_{h \to 0} \cos(h) + \cos(c) \lim_{h \to 0} \sin(h) \\
&= \sin(c)(1) + \cos(c)(0) \\
&= \sin(c)
\end{aligned}$$

and

$$\begin{aligned}\lim_{x\to c}\cos(x) &= \lim_{h\to 0}\cos(c+h)\\ &= \lim_{h\to 0}(\cos(c)\cos(h)-\sin(c)\sin(h))\\ &= \cos(c)\lim_{h\to 0}\cos(h)-\sin(c)\lim_{h\to 0}\sin(h)\cdot\\ &= \cos(c)(1)-\sin(c)(0)\\ &= \cos(c)\end{aligned}$$

Thus we have the following proposition.

Proposition The sine and cosine functions are continuous on $(-\infty,\infty)$.

The next proposition is then an immediate consequence of (2.4.8).

Proposition The tangent, cotangent, secant and cosecant functions are continuous at all points in their respective domains.

We have not yet considered the composition of continuous functions. Suppose g is continuous at c and f is continuous at $g(c)$. If $\{x_n\}$ is a sequence converging to c, then we know, since g is continuous at c, that the sequence $\{g(x_n)\}$ will converge to $g(c)$. But then, since f is continuous at $g(c)$, the sequence $\{f(g(x_n))\}$ will converge to $f(g(c))$. That is,

$$\lim_{x\to c} f\circ g(x) = \lim_{x\to c} f(g(x)) = f(g(c)) = f\circ g(c). \tag{2.4.12}$$

Hence $f\circ g$ is continuous at c.

Proposition If g is continuous at c and f is continuous at $g(c)$, then $f\circ g$ is continuous at c.

Example The function $h(t) = \cos(3t+4)$ is continuous on $(-\infty,\infty)$ since it is the composition of the functions $g(t) = 3t+4$ and $f(t) = \cos(t)$, both of which are continuous on $(-\infty,\infty)$.

Example Consider the function

$$g(t) = \frac{\sin(t^2+1)}{t}.$$

Now the numerator of g is continuous on $(-\infty,\infty)$ since it is the composition of $h(t) = t^2+1$ and $f(t) = \sin(t)$, both of which are continuous on $(-\infty,\infty)$. It follows that, since the denominator of g is continuous on $(-\infty,\infty)$, g is continuous at all points for which the denominator is not equal to zero, that is, for all $t \neq 0$. Thus g is continuous on the intervals $(-\infty,0)$ and $(0,\infty)$.

In Section 2.5 we will consider two properties of continuous functions which partially explain the important role they play in calculus.

Problems

1. Discuss the continuity of the given function at the specified point.

   (a) $f(t) = 3t^2 - 6$ at $t = 2$

   (b) $f(x) = \dfrac{2x+5}{x-16}$ at $x = 17$

   (c) $f(x) = \dfrac{2x+5}{x-16}$ at $x = 16$

   (d) $h(s) = \dfrac{s^2-1}{s+1}$ at $s = 1$

   (e) $h(s) = \dfrac{s^2-1}{s+1}$ at $s = -1$

2. Discuss the continuity of the function

$$g(t) = \begin{cases} 4t-1, & \text{if } t \le 2, \\ t+5, & \text{if } t > 2, \end{cases}$$

   at $t = 2$.

3. Discuss the continuity of the following functions.

   (a) $g(x) = 4x^{23} - x^{18} + 16x - 3$

   (b) $f(t) = \dfrac{t^2 - t - 6}{t+2}$

   (c) $g(t) = 32t - \dfrac{8}{t}$

   (d) $f(u) = \dfrac{8}{u^2-4}$

   (e) $f(t) = \sqrt{t^2-4}$

   (f) $g(x) = \dfrac{1}{\sqrt{9-x^2}}$

4. Discuss the continuity of the function

$$f(x) = \begin{cases} 3x+2, & \text{if } x < 1, \\ 3x+1, & \text{if } x \ge 1. \end{cases}$$

5. Discuss the continuity of the function

$$h(z) = \begin{cases} z^2-1, & \text{if } z \le -1, \\ z-1, & \text{if } z > -1. \end{cases}$$

6. The function

$$f(t) = \frac{t^2 - 7t + 12}{t-4}$$

   is not continuous at $t = 4$. Is this discontinuity removable? If it is, define a new function g which agrees with f whenever $t \neq 4$, but is continuous at 4.

7. The function

$$f(t) = \frac{t^2 - 7t + 12}{t-5}$$

   is not continuous at $t = 5$. Is this discontinuity removable? If it is, define a new function g which agrees with f whenever $t \neq 5$, but is continuous at 5.

8. Explain why $g(x) = x^2 \sin(x^2 + 1)$ is continuous on $(-\infty, \infty)$.
9. Recall that the Heaviside function is defined by

$$H(t) = \begin{cases} 0, & \text{if } t < 0, \\ 1, & \text{if } t \geq 0. \end{cases}$$

   (a) Discuss the continuity of $f(t) = H(t^2 + 1)$. Graph f on the interval $[-5, 5]$.
   (b) Discuss the continuity of $g(t) = H(t^2 - 1)$. Graph g on the interval $[-5, 5]$.
   (c) Discuss the continuity of $h(t) = H(\sin(\pi t))$. Graph h on the interval $[-5, 5]$.
10. Discuss the continuity of $f(x) = \lfloor x \rfloor$ and $g(x) = \lceil x \rceil$.
11. Discuss the continuity of $f(x) = \lfloor \sin(x) \rfloor$ and $g(x) = \lceil \sin(x) \rceil$.