

*Difference Equations  
to  
Differential Equations*

Section 1.3  
The Sum of a Sequence

This section considers the problem of adding together the terms of a sequence. Of course, this is a problem only if more than a finite number of terms of the sequence are nonzero. In this case, we must decide what it means to add together an infinite number of nonzero numbers. The first example shows how a relatively simple question may lead to such infinite summations.

**Example** Suppose a game is played in which a fair coin is tossed until the first time a head appears. What is the probability that a head appears for the first time on an even-numbered toss? To solve this problem, we first need to determine the probability of obtaining a head for the first time on any given even numbered toss, and then we need to add all these probabilities together. Let  $P_n$  denote the probability that the first head appears on the  $n$ th toss,  $n = 1, 2, 3, \dots$ . Then, since the coin is assumed to be fair,

$$P_1 = \frac{1}{2}.$$

Now in order to get a head for the first time on the second toss, we must toss a tail on the first toss and then follow that with a head on the second toss. Since one-half of all first tosses will be tails and then one-half of those tosses will be followed by a second toss of heads, we should have

$$P_2 = \left(\frac{1}{2}\right) \left(\frac{1}{2}\right) = \frac{1}{4}.$$

Similarly, since one-fourth of all sequences of coin tosses will begin with two tails and then half of these sequences will have a head for the third toss, we have

$$P_3 = \left(\frac{1}{4}\right) \left(\frac{1}{2}\right) = \frac{1}{8}.$$

Continuing in this fashion, it should seem reasonable that, for any  $n = 1, 2, 3, \dots$ ,

$$P_n = \frac{1}{2^n}.$$

Hence we have a sequence of probabilities  $\{P_n\}$  for  $n = 1, 2, 3, \dots$ , and, in order to find the desired probability, we need to add up the even-numbered terms in this sequence. Namely, the probability that a head appears for the first time on an even toss is given by

$$P_2 + P_4 + P_6 + \dots = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \dots. \quad (1.3.1)$$

But this involves adding together an infinite number of nonzero values. Is this possible? Can we perform the operation of addition an infinite number of times? In this case the answer is yes, but we will need a few preliminaries before we can finish this particular example.

We begin with a definition of the sum of a sequence  $\{a_n\}$ . The idea is to create a new sequence by successively adding together the terms of the original sequence. That is, we define a new sequence  $\{s_n\}$  where  $s_n$  is the sum of the first  $n$  terms of the original sequence. If

$$\lim_{n \rightarrow \infty} s_n$$

exists, then this indicates that, as we add together more and more terms of  $\{a_n\}$ , the resulting sums approach a limiting value. It is then reasonable to call this limiting value the sum of the sequence. For example, if

$$a_n = \frac{1}{2^n}$$

for  $n = 1, 2, 3, \dots$ , then we would have

$$\begin{aligned} s_1 &= \frac{1}{2}, \\ s_2 &= \frac{1}{2} + \frac{1}{4} = \frac{3}{4}, \\ s_3 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}, \\ s_4 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}, \end{aligned}$$

and so on. If you plot these points on the real line, you may think of starting at  $\frac{1}{2}$ , moving  $\frac{1}{2}$  the distance to 1 to plot the next point, then  $\frac{1}{2}$  the remaining distance to 1 to plot the next point, and so on. After  $n$  points, you would be at

$$s_n = 1 - \frac{1}{2^n}. \tag{1.3.2}$$

Clearly,

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{2^n}\right) = 1,$$

and it would be reasonable to say that the sequence adds up to 1. That is,

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots = 1. \tag{1.3.3}$$

This idea is formalized in the following definition.

**Definition** Given a sequence  $\{a_n\}$ ,  $n = 1, 2, 3, \dots$ , we define a new sequence  $\{s_n\}$  by letting

$$s_n = a_1 + a_2 + \dots + a_n \quad (1.3.4)$$

for  $n = 1, 2, 3, \dots$ . If the sequence  $\{s_n\}$  converges, then we call

$$s = \lim_{n \rightarrow \infty} s_n$$

the *sum* of the sequence  $\{a_n\}$ . The sequence  $\{s_n\}$  is called an *infinite series* and an individual term  $s_n$  of this sequence is called a *partial sum* of the sequence  $\{a_n\}$ .

Note that we have assumed that the first term in the sequence  $\{a_n\}$  in the definition is  $a_1$ . The sequence could just as well start with any other integer index, in which case the sequence of partial sums  $\{s_n\}$  would start with the same index. For example, if the first term of the sequence is  $a_0$ , then the first partial sum is  $s_0$ .

Since summations involving an infinite number, or even a large finite number, of terms are cumbersome to write using the standard plus sign of addition,  $\Sigma$  (the capital Greek *sigma*) is used to denote the process of summation. In particular, we would write

$$s_n = \sum_{j=1}^n a_j = a_1 + a_2 + \dots + a_n \quad (1.3.5)$$

and

$$s = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \sum_{j=1}^n a_j = \lim_{n \rightarrow \infty} (a_1 + a_2 + \dots + a_n). \quad (1.3.6)$$

Since (1.3.6) is what we mean by an infinite sum, we will in fact write

$$s = \sum_{j=1}^{\infty} a_j = a_1 + a_2 + \dots + a_n + \dots \quad (1.3.7)$$

For example, in this notation, we may restate our earlier results as

$$\sum_{n=1}^n \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}.$$

and

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots + \frac{1}{2^n} + \dots = 1.$$

We should note that, since the sum of a sequence is the limit of another sequence, and not all sequences have limits, there are sequences which do not have sums. For example, the sequence with terms  $a_n = 1$  for  $n = 1, 2, 3, \dots$  does not have a sum since

$$s_n = \underbrace{1 + 1 + \dots + 1}_{n \text{ times}} = n,$$

from which it follows that

$$\lim_{n \rightarrow \infty} s_n = \infty.$$

For another example, the sequence  $\{(-1)^n\}$ ,  $n = 0, 1, 2, \dots$ , does not have a sum since

$$s_n = \begin{cases} 1, & \text{if } n = 0, 2, 4, \dots, \\ 0, & \text{if } n = 1, 3, 5, \dots, \end{cases}$$

a sequence which clearly does not have a limit.

In general it may be difficult to determine the sum of a sequence; in fact, it may be difficult to determine even if the sequence has a sum. We will return to this problem in Chapter 5 when we have more tools at our disposal, as well as more motivation for studying infinite series. For now we will look at an important class of sequences for which the sum is determined with relative ease. These are the sequences for which the terms are in geometric progression; that is, sequences for which successive terms have a common ratio. We call the infinite series which corresponds to such a sequence a *geometric series*.

### Geometric series

Suppose  $\{a_n\}$  is a sequence with  $a_n = cr^{n-1}$ , where  $c \neq 0$  and  $r$  are constants and  $n = 1, 2, 3, \dots$ . Then the partial sums are

$$\begin{aligned} s_n &= a_1 + a_2 + a_3 + \dots + a_n \\ &= c + cr + cr^2 + \dots + cr^{n-1} \\ &= c(1 + r + r^2 + \dots + r^{n-1}). \end{aligned}$$

If  $r = 1$ ,  $s_n = nc$  and so  $\{s_n\}$  does not converge. If  $r \neq 1$ , it is easy to see, using long division (or the derivation outlined in Problem 4), that

$$\frac{1 - r^n}{1 - r} = 1 + r + r^2 + \dots + r^{n-1}. \quad (1.3.8)$$

Hence, if  $r \neq 1$ ,

$$s_n = \frac{c(1 - r^n)}{1 - r}. \quad (1.3.9)$$

From (1.3.9), it is clear that  $\{s_n\}$  does not converge if  $|r| \geq 1$ . But if  $-1 < r < 1$ , then

$$\lim_{n \rightarrow \infty} r^{n-1} = 0,$$

and so the sequence has the sum

$$s = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \frac{c(1 - r^n)}{1 - r} = \frac{c}{1 - r}. \quad (1.3.10)$$

That is, we have now seen that

$$\sum_{n=1}^{\infty} cr^{n-1} = \frac{c}{1 - r} \quad (1.3.11)$$

whenever  $-1 < r < 1$ .

**Example** We have, using (1.3.11) with  $c = 1$  and  $r = \frac{1}{2}$ ,

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \frac{1}{1 - \frac{1}{2}} = 2.$$

Note that this agrees with our previous result that

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1.$$

**Example** We have

$$\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n = \sum_{n=1}^{\infty} \frac{2}{3} \left(\frac{2}{3}\right)^{n-1} = \frac{\frac{2}{3}}{1 - \frac{2}{3}} = 2.$$

**Example** We have

$$\sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n = \frac{1}{1 - \frac{1}{5}} = \frac{5}{4}.$$

Note that in this example the sum starts with  $n = 0$  instead of  $n = 1$  as in (1.3.11). However, it is the initial power of  $r$  in (1.3.11) that is important, not how we write the index. Hence, the sum in this example could be written equally well as

$$\sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^{n-1},$$

or

$$\sum_{n=2}^{\infty} \left(\frac{1}{5}\right)^{n-2},$$

or

$$\sum_{n=100}^{\infty} \left(\frac{1}{5}\right)^{n-100},$$

as well as many other ways. The key in applying (1.3.11) is that we identify  $c$  and  $r$  so that the first term in the sum is  $cr^0 = c$ .

**Example** We have

$$\sum_{n=2}^{\infty} 4(0.34)^n = \sum_{n=2}^{\infty} 4(0.34)^2(0.34)^{n-2} = \frac{4(0.34)^2}{1 - 0.34} = 0.7006,$$

where we have used (1.3.11) with  $c = 4(0.34)^2$  and  $r = 0.34$ .

We are now in a position to compute the sum in (1.3.1), and hence complete our first example

**Example** Let  $P$  be the probability that, when tossing a fair coin repeatedly, the first head appears on an even toss. Then we have seen that

$$P = P_2 + P_4 + P_6 + \cdots = \sum_{n=1}^{\infty} P_{2n},$$

where

$$P_{2n} = \left(\frac{1}{2}\right)^{2n} = \left(\left(\frac{1}{2}\right)^2\right)^n = \left(\frac{1}{4}\right)^n$$

for  $n = 1, 2, 3, \dots$ . Thus

$$P = \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n = \sum_{n=1}^{\infty} \frac{1}{4} \left(\frac{1}{4}\right)^{n-1} = \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{1}{3}.$$

**Example** Economists often talk about the *multiplier effect* of an infusion of money into an economy which results in new spending many times greater than the original amount spent. This is a consequence of the recipients of the money spending a certain percentage of their new money, the recipients of this spending again spending a certain percentage of their gain, and so on. For example, suppose the government spends three million dollars, and suppose that at each stage the recipients spend 90% of the money they receive. Then the first recipients spend  $(3)(0.9) = 2.7$  million dollars, the second recipients spend

$$(3)(0.9)(0.9) = (3)(0.9)^2 = 2.43$$

million dollars (that is, 90% of the 2.7 million spent by the first recipients), the third recipients spend

$$(3)(0.9^2)(0.9) = (3)(0.9)^3 = 2.187$$

million dollars, and so on. If we denote the total amount of spending after  $n$  transactions by  $S_n$ , then, in millions of dollars,

$$S_1 = 3,$$

$$S_2 = 3 + 3(0.9),$$

$$S_3 = 3 + 3(0.9) + 3(0.9)^2,$$

$$S_4 = 3 + 3(0.9) + 3(0.9)^2 + 3(0.9)^3,$$

and, in general,

$$S_n = 3 + 3(0.9) + 3(0.9)^2 + \cdots + 3(0.9)^{n-1}$$

for  $n = 1, 2, 3, \dots$ . Although in actuality there will be only a finite number of transactions, we can see that as  $n$  increases the total spending will approach the sum

$$S = \sum_{n=1}^{\infty} 3(0.9)^{n-1} = \frac{3}{1-0.9} = 30$$

million dollars. Thus the initial governmental expenditure of three million dollars results in approximately 30 million dollars, 10 times the initial amount, in new spending in the economy. This partially explains why deficit spending by the government in depressed times can be far more beneficial to the economy than the actual amount spent, and why such spending during other times can be highly inflationary.

**Example** This example involves slightly more complicated probabilistic reasoning, as well as some additional algebraic simplification, before the problem is reduced to the summation of a geometric series. Suppose that a certain female animal has a 10% chance of dying during any given year of her life. Moreover, suppose the animal does not reproduce during her first year of life, but every year after has a 20% chance of successfully reproducing. What is the probability that this animal has offspring before dying?

Let  $P$  be the probability that the animal has offspring before dying and let  $P_n$  be the probability that the animal successfully reproduces for the first time in its  $n$ th year. Then

$$P = \sum_{n=1}^{\infty} P_n.$$

Note that our sum extends to infinity even though in reality it is highly unlikely that any such animal would live even to an age of 100 years. We do this because the model we are using, as with all mathematical models, is an idealization of the real situation. In this case, by assuming that a given animal of this species has a constant 10% chance of dying in any given year, we have implicitly assumed that there is no fixed upper bound to its life-span. Put another way, we have assigned a positive probability to an animal's living for, as an example, 1000 years, although this probability is very small (namely,  $0.9^{1000} \approx 1.748 \times 10^{-46}$ ) and, hence, is not actually ever going to happen.

Since we have assumed that these animals cannot reproduce in their first year of life, we have  $P_1 = 0$ . To compute  $P_2$ , we note that 90% of all such females will live through their first year and that 20% of these will then have offspring successfully. Hence the proportion of females that successfully reproduce for the first time in their second year is

$$P_2 = (0.9)(0.2).$$

To compute  $P_3$ , first we note that the proportion of females living until their third year will be  $(0.9)(0.9)$  (that is, 90% of the 90% who lived through their first year). Now 80% of these will not have produced offspring successfully in their second year, so the proportion of females who reach their third year without having reproduced is  $(0.9)^2(0.8)$ . Finally, 20% of these will have success in reproducing in their third year. Thus

$$P_3 = (0.9)^2(0.8)(0.2).$$

Similar reasoning yields

$$P_4 = (0.9)^3(0.8)^2(0.2)$$

(that is, this represents a female who has lived through three years, did not reproduce in either her second or third year, but did have offspring in her fourth year) and, in general,

$$P_n = (0.9)^{n-1}(0.8)^{n-2}(0.2)$$

for  $n = 2, 3, 4, \dots$ . Hence

$$\begin{aligned} P &= \sum_{n=1}^{\infty} P_n \\ &= \sum_{n=2}^{\infty} (0.9)^{n-1}(0.8)^{n-2}(0.2) \\ &= \sum_{n=2}^{\infty} (0.2)(0.9)(0.9)^{n-2}(0.8)^{n-2} \\ &= \sum_{n=2}^{\infty} (0.18)(0.72)^{n-2} \\ &= \frac{0.18}{1 - 0.72} \\ &= 0.6429, \end{aligned}$$

where the answer has been rounded to four decimal places. Thus we conclude that a given female of this species has just over a 64% chance of reproducing during her lifetime.

### The harmonic series

It may happen that a sequence  $\{a_n\}$  does not have a sum even though

$$\lim_{n \rightarrow \infty} a_n = 0.$$

One important example of this behavior is provided by the sequence  $\{a_n\}$  with

$$a_n = \frac{1}{n}$$

for  $n = 1, 2, 3, \dots$ . The resulting infinite series with  $n$ th partial sum given by

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}. \quad (1.3.12)$$

is called the *harmonic series*. Since

$$s_{n+1} = 1 + \frac{1}{2} + \cdots + \frac{1}{n} + \frac{1}{n+1} = s_n + \frac{1}{n+1} > s_n, \quad (1.3.13)$$



the sequence  $\{s_n\}$  is monotone increasing. Hence, by the Monotone Sequence Theorem,  $\{s_n\}$  either converges or diverges to infinity. Now

$$\begin{aligned} s_1 &= 1, \\ s_2 &= 1 + \frac{1}{2}, \\ s_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{1}{2} + \frac{1}{2} = 1 + 2\left(\frac{1}{2}\right), \\ s_8 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + 3\left(\frac{1}{2}\right), \end{aligned}$$

and

$$s_{16} = s_8 + \sum_{j=9}^{16} \frac{1}{j} > s_8 + \sum_{j=9}^{16} \frac{1}{16} > 1 + 3\left(\frac{1}{2}\right) + \frac{8}{16} = 1 + 4\left(\frac{1}{2}\right).$$

Continuing in this pattern, we can see that, in general,

$$s_{2^m} > 1 + \frac{m}{2} \tag{1.3.14}$$

for any  $m = 0, 1, 2, \dots$ . Thus, since  $\frac{m}{2}$  may be made arbitrarily large, the sequence  $\{s_n\}$  does not have an upper bound. Hence we must have

$$\lim_{n \rightarrow \infty} s_n = \infty, \tag{1.3.15}$$

and so the harmonic series does not have a sum.

Although the partial sums of the harmonic series diverge to infinity, they grow very slowly. For example, if  $n = 500,000,000$ , then  $s_n$  is between 20 and 21. That is,

$$20 < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{500,000,000} < 21. \tag{1.3.16}$$

## Problems

1. Find the sum of each of the following infinite series which has a sum.

(a)  $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^{n-1}$

(b)  $\sum_{n=1}^{\infty} 4(0.21)^{n-1}$

(c)  $\sum_{n=1}^{\infty} \frac{2}{5^n}$

(d)  $\sum_{n=0}^{\infty} \frac{2}{7^n}$

(e)  $\sum_{n=1}^{\infty} 7\left(\frac{1}{3}\right)^n \left(\frac{2}{5}\right)^{n-1}$

(f)  $\sum_{n=3}^{\infty} \left(\frac{2}{3}\right)^n$

$$\begin{array}{ll}
 \text{(g)} \sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^{n-1} & \text{(h)} \sum_{n=1}^{\infty} 0.99999^n \\
 \text{(i)} \sum_{n=1}^{\infty} 1.00001^n & \text{(j)} \sum_{n=30}^{\infty} 5 \left(\frac{3^n}{7^{n-1}}\right) \\
 \text{(k)} \sum_{n=100}^{\infty} \left(\frac{91}{89}\right)^{n-1} & \text{(l)} \sum_{n=1}^{\infty} \left(\frac{\pi}{4}\right)^n \\
 \text{(m)} \sum_{n=1}^{\infty} \sin(\pi n) & \text{(n)} \sum_{n=1}^{\infty} \cos(\pi n)
 \end{array}$$

2. Consider the infinite series with  $n$ th partial sum

$$s_n = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \cdots + \frac{1}{n!} = \sum_{j=0}^{\infty} \frac{1}{j!}.$$

Note that, by definition,  $0! = 1$ .

(a) Show that  $s_n < 3$  for all values of  $n$ . Hint: Note that

$$n! = (1)(2)(3) \cdots (n) \geq (1)(2)(2) \cdots (2) = 2^{n-1}$$

for  $n = 1, 2, 3, \dots$

(b) Combine (a) with the fact that  $s_{n+1} > s_n$  for all  $n$  to conclude that

$$\sum_{j=1}^{\infty} \frac{1}{j!}$$

exists and is less than 3.

(c) In fact,

$$\sum_{j=1}^{\infty} \frac{1}{j!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots$$

is a well known irrational number. Add up a sufficient number of terms to enable you to guess the value of the sum. How many terms did it take?

(d) How many terms are necessary to obtain a partial sum that is within 0.000001 of the sum?

3. The sum

$$4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 4 \left( 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \right)$$

is a well known irrational number.

(a) Add up a sufficient number of terms to enable you to guess the value of the sum. How many terms did it take?

- (b) How many terms are necessary to obtain a partial sum that is within 0.01 of the sum?
4. This problem outlines an alternative method for deriving the result of (1.3.8). Suppose  $r \neq 1$ . For  $n = 1, 2, 3, \dots$ , let

$$s_n = 1 + r + r^2 + \dots + r^{n-1}.$$

Show that  $s_n - rs_n = 1 - r^n$  and conclude that

$$s_n = \frac{1 - r^n}{1 - r}.$$

5. Using the model we used for the multiplier effect, find the total amount of new spending resulting from each of the following.
- (a) The government spends 2 billion dollars; each recipient spends 80% of what he or she receives.
  - (b) The government spends 250 million dollars; each recipient spends 95% of what he or she receives.
  - (c) The government spends  $A$  dollars; each recipient spends  $100r\%$ ,  $0 < r < 1$ , of what he or she receives.
6. Government regulations specify that a bank may not loan 100% of its deposits; the bank must keep a certain percentage of its deposits in reserve. For example, if a bank must keep 15% of its deposits in reserve, then it may loan out \$850 from a \$1000 deposit. Typically, this \$850 will again be deposited in a bank, and that bank may loan out 85% of it. Again, this money will be deposited and 85% of it given out in loans. As this will continue indefinitely, the multiplier effect comes into play and the total amount of money in all the deposits resulting from the initial \$1000 deposit can be computed in the same manner as in our example.
- (a) Compute the total amount of the deposits resulting from the initial \$1000 deposit.
  - (b) How would the answer in (a) change if the reserve rate was changed from 15% to 20%?
  - (c) How would the answer in (a) change if the reserve rate was changed from 15% to 10%?
7. A ball is dropped from a height of 10 meters. Suppose that every time it strikes the ground, it bounces back to a height which is 75% of the height of the previous bounce. Assuming an infinite number of bounces (again, an idealized mathematical model), how far does the ball travel before it comes to rest? What would happen if it rebounded to only 25% of its initial height?
8. Suppose the animal in the our final example above could not produce offspring for its first 3 years of life. How would this change the probability of a female's reproducing before dying?

9. Suppose the animal in our final example above has only an 80% chance of living through a given year. How does this change the probability of a female's reproducing before dying?
10. Suppose a female animal of the type discussed in the final example above has a  $100r\%$ ,  $0 \leq r \leq 1$ , chance of reproducing each year after its first year.
- Find the probability  $P$  of a female's reproducing before dying.
  - Plot  $P$  as a function of  $r$  for  $0 \leq r \leq 1$ .
  - Find the value of  $r$  for which  $P = 0.5$ .
11. How many terms of the harmonic series are needed to obtain a partial sum larger than 5? How many terms are needed to obtain a partial sum larger than 10?
12. Plot the points  $(n, s_n)$ , where  $s_n$  is the  $n$ th partial sum of the harmonic series, for  $n = 1, 2, 3, \dots, 1000$ . What does this show you about the rate of growth of the partial sums?
13. The first example of this section is a particular case of the more general problem of computing probabilities associated with the waiting time for some event to occur. As another example, suppose that an electronic switch works with probability  $p$  and fails with probability  $q = 1 - p$ . Then, using reasoning analogous to that used in the coin tossing example, the probability that the first failure will occur on the  $n$ th use of the switch is  $p^{n-1}q$ ,  $n = 1, 2, 3, \dots$
- Can you justify this probability?
  - The reliability of the switch is given by the function

$$\begin{aligned} R(n) &= \text{probability that the switch does not fail until after the } n\text{th use} \\ &= \sum_{j=n+1}^{\infty} p^{j-1}q \end{aligned}$$

Show that  $R(n) = p^n$ ,  $n = 1, 2, 3, \dots$

- Find a way to show that  $R(n) = p^n$  directly without using an infinite series.